Lipschitz-free space over countable compact metric spaces

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I. Introduction

Definitions

Let (M,d) be a pointed metric space and $Lip_0(M) = \{f : M \to \mathbb{R} \text{ Lipschitz} \,;\, f(0) = 0\}$. Equipped with the Lipschitz norm

$$\|f\|_L = \sup_{x \neq y} \frac{f(x)-f(y)}{d(x,y)}, \quad f \in Lip_0(M)$$

this space is a Banach space.
Moreover its unit ball is compact for the pointwise topology, then it is a dual space.

Definition
The Lipschitz-free space over M, denoted $\mathcal{F}(M)$, is the predual of $Lip_0(M)$. It is defined by $\mathcal{F}(M) = \overline{\text{vect}}\{\delta_x \,;\, x \in M\} \subset Lip_0(M)$.

Very little is known about the linear structure of Lipschitz-free spaces. For instance we know that $\mathcal{F}(\mathbb{R})$ is isomorphically isometric to L_1, but $\mathcal{F}(\mathbb{R}^2)$ is not isomorphic to any subspace of L_1 (Naor-Schechtman).

Lipschitz-free spaces and BAP

Definition: *Let X be a Banach space.*

- *X has approximation property (AP) if $\forall K \subset X$ compact, $\forall \varepsilon > 0$, there is an operator $T : X \to X$ of finite rank, such that $\forall x \in K$, $\|Tx - x\| < \varepsilon$.*
- *X has λ-bounded approximation property (λ-BAP), $\lambda \geqslant 1$, if $\forall K \subset X$ compact, $\forall \varepsilon > 0$ there is $T : X \to X$ of finite rank, such that $\forall x \in K$, $\|Tx - x\| < \varepsilon$ and $\|T\| \leqslant \lambda$.*
- *X has metric approximation property (MAP) if it has 1-BAP.*

Theorem[Godefroy, Kalton, 2003]: *For a Banach space, X has λ-BAP if and only if $\mathcal{F}(X)$ has λ-BAP.*
Theorem[Lancien, Pernecká]: *The Lipschitz-free space over a doubling metric space has BAP. (M doubling if there is a constant D so that every ball of radius R can be covered with D ball of radius $R/2$)*
However,

Theorem [Godefroy, Ozawa]
There exists a compact metric space K such that $\mathcal{F}(K)$ fails AP.

Question: Does every countable compact metric space have BAP ?

II. Kalton decomposition

For a pointed metric space M and $n \in \mathbb{Z}$, let

$$M_n = \{x \in M \mid d(0,x) \leqslant 2^n\}$$
$$O_n = \{x \in M \mid d(0,x) < 2^n\}$$
$$A_n = M_{n+1} \backslash O_{-n-1} \cup \{0\}$$

and

$$T_n\delta(x) = \begin{cases} 0 & , x \in M_{n-1} \\ (\log_2 d(0,x) - (n-1))\,\delta(x) & , x \in M_n \backslash M_{n-1} \\ (n+1-\log_2 d(0,x))\,\delta(x) & , x \in M_{n+1} \backslash M_n \\ 0 & , x \notin M_{n+1} \end{cases}$$

Theorem [Kalton, 2004]
For every γ in $\mathcal{F}(M)$, $\gamma = \sum_{n \in \mathbb{Z}} T_n\gamma$ unconditionnally and $\sum_{n \in \mathbb{Z}} \|T_n\gamma\| \leqslant 72\|\gamma\|$.

Consequently

- $\mathbf{M' = \{0\}}$
  We denote, $\forall N \in \mathbb{N}$, $S_N = \sum_{n=-N}^{n=N} T_n : \mathcal{F}(M) \to \mathcal{F}(A_N)$, then the sequence of operators $(S_N)_N$ verifies:
  - the rank of S_N is finite
  - $\forall \gamma \in \mathcal{F}(M)$, $S_N\gamma \to \gamma$
  - $\forall N \in \mathbb{N}$, $\|S_N\| \leqslant 72$

  That is: $\mathcal{F}(M)$ has 72-BAP.
- $\mathbf{M'}$ **finite**: $M' = \{a_1, ..., a_k\}$
  Then $\mathcal{F}(M) \simeq \left(\oplus_{i=1}^k \mathcal{F}(F_i)\right)$, with $F_i' = \{a_i\}$. So $\forall i$, $\mathcal{F}(F_i)$ has 72-BAP.
  And BAP is stable under finite ℓ_1-sum and isomorphism.

Question: Does $M^{(2)} = \{0\}$ imply $\mathcal{F}(M)$ has BAP ?

$M^{(2)} = \{0\} \Rightarrow \mathcal{F}(M)$ has BAP ?

The first idea is to use again the decomposition :

$M^{(2)} = \{0\} \Rightarrow A_n'$ finite, $\forall n \in \mathbb{Z}$. So $\mathcal{F}(A_n')$ has BAP :
$\forall n \in \mathbb{Z}$, $\exists L_j^n : \mathcal{F}(A_n) \to \mathcal{F}(A_n)$ finite rank operators,

- $\forall \gamma \in \mathcal{F}(A_n)$, $\lim_j L_j^n\gamma = \gamma$
- $\forall j$, $\|L_j^n\| \leqslant C_n$

Moreover the decomposition gives :
$S_n : \mathcal{F}(M) \to \mathcal{F}(A_n)$ s.t.

- $\forall \mu \in \mathcal{F}(M)$, $\lim_n S_n\mu = \mu$
- $\forall n$, $\|S_n\| \leqslant 72$

Combining these operators we obtain $R_{j,n} = L_j^n \circ S_n : \mathcal{F}(M) \to \mathcal{F}(M)$ finite rank operator s.t.

- $\forall \mu \in \mathcal{F}(M)$, $\lim_{j,n} R_{j,n}\mu = \mu$
- $\forall n, j$, $\|R_{n,j}\| \leqslant 72C_n$

Problem: *The constant depends on n...*
Question: Does there exist a universal constant C so that M' finite $\Rightarrow \mathcal{F}(M)$ has C-BAP ?

III. Duality

Theorem[Grothendieck]: *X separable Banach space, isometric to a dual space. If X has AP, then X has MAP.*
In order to use that theorem of Grothendieck to obtain $C = 1$, the idea is to see Lipschitz-free spaces over countable compact metric spaces as dual spaces. Let us define the space of which $\mathcal{F}(M)$ will be the dual space:

Definition
Let $f \in Lip_0(M)$. We say that f is in $lip_0(M)$ if $\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$$d(x,y) < \delta \Rightarrow |f(x) - f(y)| \leqslant \varepsilon d(x,y)$$

To prove that the dual space of $lip_0(M)$ is $\mathcal{F}(M)$ we will use a theorem due to Petunīn and Plīčko:
Theorem: *Let X be a separable Banach space and $S \subset X^*$ closed such that : $S \subset NA(X)$ and S is separating. Then $S^* \equiv X$.*
We know that, in the case of compact spaces, $lip_0(M)$ is a subset of $NA(\mathcal{F}(M))$ and it is separating if and only if it separates points uniformly.
Definition: *$f \in lip_0(M)$ separates points uniformly if*

$$\exists c \geqslant 1 : \forall x, y \in M, \exists h = h_{x,y} \in lip_0(M), \|h\|_L \leqslant c \text{ and } |h(x) - h(y)| = d(x,y)$$

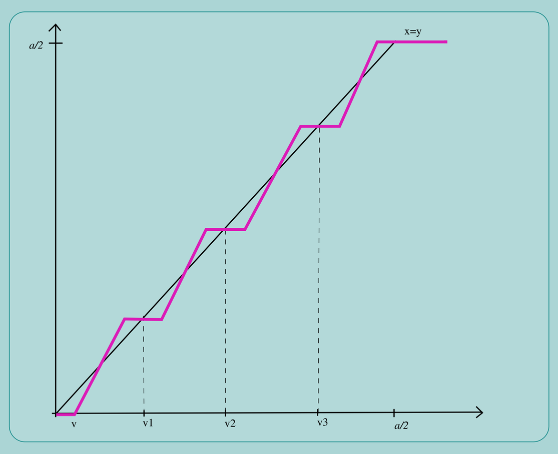
Theorem
For every M countable compact metric space, $\mathcal{F}(M)$ is a dual space.

III. Duality

Idea of the proof

Assume $\mathbf{M'}$ **finite**.
Let $x, y \in M$ and set $a = d(x,y)$. Then $M' \cap B(x, a/2) \backslash \{x\} = \{y_1, ..., y_n\}$.
If $a_i = d(x, y_i)$, $1 \leqslant i \leqslant n$, we can write $\{a_i, \, i \leqslant n\} = \{v_1 < ... < v_r\}$.
Now set $v = \min\{v_1\} \cup \{|v_i - v_{i-1}|\}$ and define $\varphi : [0, +\infty[\to [0, +\infty[$ piecewise affine :

$$\varphi(t) = \begin{cases} 0 & , t \in \left[0, \frac{v}{4}\right[:= V_0 \\ v_i & , t \in \left]v_i - \frac{v}{4}, v_i + \frac{v}{4}\right[:= V_i, \; 1 \leqslant i \leqslant r \\ \frac{a}{2} & , t \in \left]\frac{a}{2} - \frac{v}{4}, +\infty\right[:= V_{r+1} \end{cases}$$



Then if $f = d(.,x)$, the set $C = f^{-1}\left([0, +\infty[\backslash \bigcup_{i=0}^{r+1} V_i\right)$ is finite. Moreover $h = 2\left(\varphi(d(.,x)) - \varphi(d(0,x))\right)$ verifies $h(0) = 0$, $|h(x) - h(y)| = d(x,y)$ and $\|h\|_L \leqslant 4$.
If we set $\delta = \frac{1}{2}\min\{v\} \cup \{d(z,t) \,,\, z \neq t \in C\} \cup \{\text{dist}(z, K \backslash C) \,,\, z \in C\}$, we obtain that if $d(z,t) \leqslant \delta$, then $h(z) = h(t)$ and finally $h \in lip_0(M)$.

In the case $\mathbf{M^{(\alpha)}}$ **finite**, where $\alpha < \omega_1$, we construct a function constant around points of $M^{(\alpha)}$. Let C_1 be what remains. We can find $\alpha_1 < \alpha$ such that $C_1^{(\alpha_1)}$ is finite and we construct a function constant around points of $C_1^{(\alpha_1)}$.
... and so on ...
We obtain a decreasing sequence $... < \alpha_n < ... < \alpha_2 < \alpha_1 < \alpha$ of ordinals, then it stops: $\exists n \in \mathbb{N}$ such that C_n is finite. The last function we obtain is in $lip_0(M)$, separates x and y and has a norm bounded with a constant not depending in x and y.

M countable compact metric space $\Rightarrow \mathcal{F}(M)$ MAP

Corollary
For every M countable compact metric space, $\mathcal{F}(M)$ has MAP.

Idea of the proof : We proceed by induction on ordinals.
The idea is the same using in the case $M^{(2)} = \{0\}$ (which didn't work because of the dependance in n of the constant): on each step we use Kalton decomposition and Grothendieck Theorem to deduce the result from the last step.

IV. Extension of the result to proper spaces

Definition: *We say that a metric space M is proper if every closed ball is compact.*

Theorem
If M is proper and countable, then $\mathcal{F}(M)$ has MAP.

Fist we prove that this subset of $lip_0(M)$:

$$S = \left\{ f \in lip_0(M) \,;\, \lim_{d(y,o) \to +\infty} \frac{f(y)}{d(y,0)} = 0, \; \lim_{r \to +\infty} \sup_{x \neq y \notin \overline{B}(0,r)} \frac{f(y) - f(y)}{d(x,y)} = 0 \right\}$$

separates uniformly points of M. Then using Petunīn and Plīčko Theorem we have that $\mathcal{F}(M)$ is a dual space.
Finally, we can apply the last result, use Kalton decomposition and Grothendieck Theorem to obtain $\mathcal{F}(M)$ has MAP.

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