FREE SPACES OVER ULTRAMETRIC SPACES ARE NEVER ISOMETRIC TO ℓ_1

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Abstract. We show that the Lipschitz free space over an ultrametric space is not isometric to $\ell_1(\Gamma)$ for any set Γ.

In this short note we give a strong negative answer to a question by M. Cúth and M. Doucha [2]. Among other results, they proved that if M is a separable (infinite) ultrametric space, then $\mathcal{F}(M)$ is isomorphic to ℓ_1 and they asked whether $\mathcal{F}(M)$ is always isometric to ℓ_1. Our main result shows that it is actually never the case. We have been informed by M. Cúth and M. Doucha that they independently obtained the same result. Their approach is different from ours and appears in the updated version of [2].

We recall that a metric space (M, d) is *ultrametric* if $d(x, y) \leq \max\{d(x, z), d(y, z)\}$ for any $x, y, z \in M$. As usual, for a metric space M with a distinguished point $0 \in M$, the *Lipschitz free space* $\mathcal{F}(M)$ is the norm-closed linear span of $\{\delta_x : x \in M\}$ in the space $\mathrm{Lip}_0(M)^*$, where the Banach space $\mathrm{Lip}_0(M) = \{f \in \mathbb{R}^M : f \text{ Lipschitz}, f(0) = 0\}$ is equipped with the norm $\|f\|_L := \sup\left\{\dfrac{f(x) - f(y)}{d(x, y)} : x \neq y\right\}$. It is well known that $\mathcal{F}(M)^* = \mathrm{Lip}_0(M)$ isometrically, which is the only property that we will need. More about the very interesting class of Lipschitz free spaces can be found in [3].

Theorem 1. *Let (M, d) be an ultrametric space of cardinality larger than 3. Then $\mathcal{F}(M)$ is not isometric to $\ell_1(\Gamma)$ for any set Γ.*

Proposition 2 which is stated after the proof of Theorem 1 reveals some more geometric information about the ball of $\mathrm{Lip}_0(M)$ and provides an alternative proof for the case when M is finite.

Proof of Theorem 1. We will find two extreme points of the unit ball $B_{Lip_0(M)}$ of $Lip_0(M)$ at distance less than 2, which is not possible in the unit ball of $\ell_\infty(\Gamma)$.

Let $x_0 = 0, x_1, y_0$ in M and assume that $d(0, y_0) = d(x_1, y_0) \geq d(0, x_1)$. Set

$$\begin{aligned} A &= \{x \in M \ : \ d(x, y_0) = d(0, y_0)\} \\ C &= \{x \in M \ : \ d(x, y_0) > d(0, y_0))\} \\ B &= \{y \in M \ : \ d(y, y_0) < d(0, y_0)\} \end{aligned}$$

Note that, because M is ultrametric, if $x \in A, y \in B$, then $d(x, y) = d(0, y_0)$, and if $x \in A, y \in B, z \in C$, then $d(x, z) = d(z, y) = d(0, z) = d(z, y_0)$.

Let $f = d(\cdot, 0) \in B_{Lip_0(M)}$ and define $g \in Lip_0(M)$ as follows:

$$g(y) = \begin{cases} f(y) = d(0,y), & y \in A \cup C \\ \sup\limits_{z \in A\cup C} d(0,z) - d(z,y) = \sup\limits_{z \in A\cup C} d(0,z) - d(z,y_0), & y \in B \end{cases}$$

Note that for $y \in B$, $d(0,x_1) - d(0,y_0) \leq g(y) \leq 0$.

We prove that g belongs to $B_{Lip_0(M)}$.

- If $x, y \in B$, $g(x) - g(y) = 0$
- If $x, y \in A \cup C$, $|g(x) - g(y)| = |d(0,y) - d(0,x)| \leq d(x,y)$.
- Let $x \in A \cup C$ and $y \in B$. First if $C \neq \emptyset$, $|g(x) - g(y)| = d(x,0) \leq d(x,y)$. Now assume $C = \emptyset$:
$$\frac{|g(x) - g(y)|}{d(x,y)} = \frac{1}{d(0,y_0)}\left(d(0,x) + d(0,y_0) - \sup_{z\in A}(d(0,z))\right) \leq \frac{d(0,y_0)}{d(0,y_0)}, \quad \text{since } x \in A.$$

Therefore $\|g\|_L \leq 1$ and $f, g \in B_{Lip_0(M)}$.

It is clear that f is an extreme point of $B_{Lip_0(M)}$. To see the same for g, we write $g = \dfrac{v+w}{2}$, with $v, w \in B_{Lip_0(M)}$. Thus,

- $v(0) = w(0) = 0$
- for every $z \in A \cup C$, we have: $g(z) = d(0,z)$, $v(z) \leq d(0,z)$, $w(z) \leq d(0,z)$ and $d(0,z) = \dfrac{v(z)+w(z)}{2}$. Thus $v(z) = w(z) = g(z)$.
- let $y \in B$. For every $z \in A \cup C$,
$$|d(0,z) - v(y)| = |g(z) - v(y)| = |v(z) - v(y)| \leq d(z,y)$$
Hence
$$g(y) = \sup_{z \in A\cup C}(d(0,z) - d(z,y)) \leq v(y).$$
The same holds for w so we get that $v = w = g$. Hence g is an extreme point of $B_{Lip_0(M)}$.

To conclude we need to compute the distance between f and g. Note that if $x \in A \cup C$, then $g(x) - f(x) = 0$ and if $y \in B$, then $d(0,x_1) - 2d(0,y_0) \leq g(y) - f(y) \leq -d(0,y_0)$.

$$\begin{aligned} \|g - f\|_L &= \sup_{x\neq y} \frac{|g(x) - f(x) - g(y) + f(y)|}{d(x,y)} \\ &= \max\left\{ \sup_{\substack{x\in A\cup C \\ y \in B}} \frac{|g(x) - f(x) - g(y) + f(y)|}{d(x,y)}, \sup_{x\neq y \in B} \frac{|g(x) - f(x) - g(y) + f(y)|}{d(x,y)} \right\} \\ &\leq \max\left\{ \sup_{x\in A\cup C} \frac{2d(0,y_0) - d(0,x_1)}{d(x,y_0)}, \sup_{x\neq y\in B} \frac{|d(0,x) - d(0,y)|}{d(x,y)} \right\} \\ &\leq \max\left\{ \frac{2d(0,y_0) - d(0,x_1)}{d(0,y_0)}, 1 \right\} < 2 \end{aligned}$$

Finally f and g are two extreme points of $B_{Lip_0(M)}$ which are at distance less than 2. Since the extreme points are preserved by the surjective isometries, this ball cannot be isometric to $B_{\ell_\infty(\Gamma)}$ and thus the free space over M cannot be isometric to $\ell_1(\Gamma)$. $\square$

Proposition 2. *Let $M = \{x_0, x_1, \dots, x_n\}$, $n \geq 2$, be an ultrametric space. Let $x_0 = 0$ be the distinguished point. Then $B_{\mathrm{Lip}_0(M)}$ is not the intersection of less than $2n+1$ halfspaces. Thus $\mathrm{Lip}_0(M)$ is not isometric to ℓ_∞^n and $\mathcal{F}(M)$ is not isometric to ℓ_1^n.*

Proof. The proof is based on the following modification of Lemma 2.3 in [1].

Lemma 3. *Let X be a Banach space. Let $C = \bigcap_{i=1}^n x_i^{*-1}(-\infty, 1]$ where $x_i^* \in X^*$. Let $A \subset X \setminus C$ have the following property: for every $x \neq y \in A$, we have $\dfrac{x+y}{2} \in C$. Then the cardinality $|A|$ of A is at most n.*

Proof. For $x \in A$ let $\varphi(x) := i$ for some $i \in \{1, \dots, n\}$ such that $x_i^*(x) > 1$. Since $1 \geq x_{\varphi(x)}^*\left(\dfrac{x+y}{2}\right)$ it follows that $x_{\varphi(x)}^*(y) < 1$ for every $y \in A$, $y \neq x$. Thus $\varphi(x) \neq \varphi(y)$ if $x \neq y$. So $|A| \leq n$. $\square$

It is thus sufficient to find at least $2n+1$ functions in $\mathrm{Lip}_0(M) \setminus B_{\mathrm{Lip}_0(M)}$ such that all the midpoints will lie in $B_{\mathrm{Lip}_0(M)}$. In fact, we will find for every $i \neq j \in \{0, \dots, n\}$ a function f_{ij} in the sphere of $Lip_0(M)$ such that $\frac{f_{ij}(x_k) - f_{ij}(x_l)}{d(x_k, x_l)} = 1$ if and only if $i = k$ and $j = l$. Therefore there is $\lambda > 1$ such that $A = \{\lambda f_{ij} : 0 \leq i \neq j \leq n\}$ has the desired property and $|A| = (n+1)n \geq 2n+1$ as $n \geq 2$.

Lemma 4. *Let N be a finite ultrametric space, $x \in N$. Let $f : N \setminus \{x\} \to \mathbb{R}$ be a Lipschitz function. Then f can be defined at the point x in such a way that $\left|\frac{f(x)-f(y)}{d(x,y)}\right| < \|f\|_{Lip_0(N \setminus \{x\})}$ for every $y \in N \setminus \{x\}$.*

Proof. Since N is ultrametric and finite, we can add to it a new point z with the property $d(y,x) = d(y,z) + d(z,x)$ for every $y \in N \setminus \{x\}$ and the resulting space will still be a metric space. Indeed, we define the distances between z an the original points as follows: $d(z,x) \leq \frac{1}{2}\min\{d(x,y) : y \in N \setminus \{x\}\}$ and $d(z,y) = d(x,y) - d(z,x)$ for $y \in N \setminus \{x\}$. The triangle inequality is an immediate consequence of the ultrametricity of N. Now, let us define f at the point z so that the Lipschitz constant does not increase (for example by the infimal convolution formula). It is clear that for every $y \in N \setminus \{x\}$ we have $f(z) + [-d(x,z), d(x,z)] \subset f(y) + [-d(y,x), d(y,x)]$. We define $f(x) := f(z)$. It follows that $\left|\frac{f(x)-f(y)}{d(x,y)}\right| < 1$ for all $y \in N \setminus \{x\}$. $\square$

We put $\widetilde{f}_{ij}(x_i) - \widetilde{f}_{ij}(x_j) = d(x_i, x_j)$ and we extend $\widetilde{f}_{ij}$ to M by a repeated application of the above lemma. Then we let $f_{ij}(\cdot) = \widetilde{f}_{ij}(\cdot) - \widetilde{f}_{ij}(0)$. $\square$

References

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