OBSTRUCTION TO UNIFORM OR COARSE EMBEDDABILITY INTO REFLEXIVE BANACH SPACES

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Abstract. This paper is based on the paper [11] of N. J. Kalton. The main result is that c_0 cannot be uniformly or coarsely embedded into a reflexive Banach space. In order to prove it, we will present a Ramsey type argument and Kalton's property $\mathcal{Q}$, which used together permit to rule out coarse or uniform embeddings into reflexive Banach spaces.

1. Introduction

Let (M,d), (N,δ) be metric spaces and $f : M \to N$ be any map. For $t > 0$, define

$$\varphi_f(t) = \inf\{\delta(f(x), f(y));\ d(x,y) \geq t\}$$

and

$$\omega_f(t) = \sup\{\delta(f(x), f(y));\ d(x,y) \leq t\}.$$

The map f is said to be:

- a coarse embedding if $\lim\limits_{t\to+\infty} \varphi_f(t) = +\infty$ and $\omega_f(t) < +\infty$, $\forall t > 0$. Then M coarsely embeds into N.
- a uniform embedding if $\lim\limits_{t\to 0} \omega_f(t) = 0$ and $\varphi_f(t) > 0$, $\forall t > 0$. Then M uniformly embeds into N.
- a strong uniform embedding if f is a coarse and a uniform embedding.
- a Lipschitz embedding if there exist $A, B > 0$ such that for every $x, y \in M$,

$$Ad(x,y) \leq \delta(f(x), f(y)) \leq Bd(x,y).$$

In 1974, Aharoni [1] proved that every separable metric space can be Lipschitz embedded into c_0. There exist quantitative versions of this result due to Assouad [4], Pelant [17] and finally the sharp constant of distortion is 2 and is given by Kalton and Lancien in [13]. It is an open question to know whether there exist other Banach spaces into which every separable metric spaces can be Lipschitz embedded.

This question is equivalent to the following: if c_0 Lipschitz-embeds into a Banach space, does it imply that it linearly embeds into this space? In [10] Kalton proved that there exists a Banach space into which c_0 strong uniformly embeds but does not linearly embed. More precisely, for any non trivial gauge ω and any metric space (M,d), the Lipschitz-free space over $(M, \omega \circ d)$, denoted $\mathcal{F}_\omega(M)$, is a Schur space. Now $\omega : \begin{array}{ccc} \mathbb{R}_+ & \to & \mathbb{R}_+ \\ t & \mapsto & \begin{cases} t^\alpha, & t \leq 1 \\ t, & t \geq 1 \end{cases} \end{array}$ is non trivial, thus $\mathcal{F}_\omega(c_0)$ is a Schur space. Moreover it is easy to see that the identity from $(c_0, \|\cdot\|_\infty)$ to $(c_0, \omega \circ \|\cdot\|_\infty)$ is a strong uniform embedding. It is known from [9] that $(c_0, \omega \circ \|\cdot\|_\infty)$ isometrically embeds into its Lipschitz-free space. Finally, we conclude that c_0 strong uniformly embeds into $\mathcal{F}_\omega(c_0)$, which is a Schur space, hence c_0 cannot be linearly embedded into it.

It was proved independently by Christensen [7], Mankiewicz [15] and Aronszajn [3] in the 70's that if a separable Banach space X Lipschitz embeds into a space Y with the Radon-Nikodym property, the embedding admits a point of Gâteaux-differentiability and one can deduce that X linearly embeds into Y. Thus, because every reflexive space has the RNP, it is not possible to find a reflexive Banach space which is universal for Lipschitz embeddings of separable metric space, but one can ask whether there exists a reflexive Banach space into which every separable metric space could be uniformly or coarsely embedded. Following a paper of Kalton [11] (see also [14] or [8]) we will prove that there exists no reflexive Banach space containing uniformly or coarsely the space c_0. More precisely we will define a property, failed by c_0, and prove that a Banach space failing this property cannot be uniformly or coarsely embedded into a reflexive Banach space. This implies a previous result: Mendel and Naor proved in [16] that c_0 cannot be coarsely embedded into a super-reflexive Banach space. However Baudier obtained in [5] that any Banach space without cotype contains strongly uniformly every proper metric space. In particular $\left(\oplus_{n=1}^{+\infty}\ell_\infty^n\right)_2$, which is reflexive, contains strongly uniformly every proper metric space.

Section 2 is about Ramsey theory and is devoted to the proof of a Ramsey type argument due to Kalton [11]. In section 3 we introduce the $\mathcal{Q}$-property and prove that a Banach space failing it cannot be uniformly or coarsely embedded into a reflexive Banach space. In section 4 it is proved first that a stable Banach space has the $\mathcal{Q}$-property. Then we present a theorem which permits to rule out the $\mathcal{Q}$-property and we use it to prove that the James space J and its dual fail it. To conclude this section, we focus on the space c_0 and prove that it does not have the $\mathcal{Q}$-property. Then we prove a stronger result of Kalton: c_0 cannot be uniformly or coarsely embedded into a Banach space having all its iterated duals separable. Finally in section 5, we compare the structure of the paper [11] with the proof of the fact that $\mathcal{C}[1,\omega_1]$ cannot be uniformly embedded into ℓ_∞ in [12].

2. Preliminaries: Ramsey theory and special graphs

Let $\mathbb{M}$ be an infinite subset of $\mathbb{N}$ and $k \in \mathbb{N}$. The set $G_k(\mathbb{M})$ is the set of all subsets of $\mathbb{M}$ of size k. We will write an element $\overline{n}$ of $G_k(\mathbb{M})$ as follows: $\overline{n} = \{n_1, \dots, n_k\}$, with $n_1 < \dots < n_k$.

First we state Ramsey's theorem (see [18]):

Theorem 2.1. *Let $k, r \in \mathbb{N}$ and $f : G_k(\mathbb{N}) \to \{1, \dots, r\}$ be any map. Then there exists an infinite subset $\mathbb{M}$ of $\mathbb{N}$ and $i \in \{1, \dots, r\}$ such that for every $\overline{n} \in G_k(\mathbb{M})$, $f(\overline{n}) = i$.*

It is not difficult to deduce a topological version of this result.

Corollary 2.2. *Let (K, d) be a compact metric space, $k \in \mathbb{N}$ and $f : G_k(\mathbb{N}) \to K$. Then for every $\varepsilon > 0$, there exists an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that for every $\overline{n}, \overline{m} \in G_k(\mathbb{M})$, $d(f(\overline{n}), f(\overline{m})) < \varepsilon$.*

We can think about a result as a part of Ramsey theory if for a given coloring of a mathematical object, there exists a sub-object which is monochromatic.

From now we will follow the paper of Kalton [11] (see also [14], [8]). For an infinite subset $\mathbb{M}$ of $\mathbb{N}$, endow the space $G_k(\mathbb{M})$ with the following metric d: two distinct subsets $\overline{n}, \overline{m} \in G_k(\mathbb{M})$ are said to be adjacent ($d(\overline{n}, \overline{m}) = 1$) if

$$n_1 \le m_1 \le n_2 \le \cdots \le n_k \le m_k \text{ or } m_1 \le n_1 \le m_2 \le \cdots \le m_k \le n_k.$$

We will write $\overline{n} < \overline{m}$ when $n_k < m_1$. In this case, $d(\overline{n}, \overline{m}) = k$.

We will start by a Ramsey type result which will be useful to give an obstruction to uniform and coarse embeddability into reflexive Banach spaces. Before to state it we need some tools.

Let X be a Banach space, $k \in \mathbb{N}$, $f : G_k(\mathbb{N}) \to X$ a bounded map and $\mathcal{U}$ a non-principal ultrafilter on $\mathbb{N}$. We define a bounded map $\partial_{\mathcal{U}} f : G_{k-1}(\mathbb{N}) \to X^{**}$ as follows:

$$\forall \overline{n} \in G_{k-1}(\mathbb{N}),\ \partial_{\mathcal{U}} f(\overline{n}) = w^*\text{-}\lim_{n_k \in \mathcal{U}} f(n_1, \ldots, n_{k-1}, n_k).$$

We can iterate this procedure for $1 \le r \le k$: $\partial_{\mathcal{U}}^r f : G_{k-r}(\mathbb{N}) \to X^{(2r)}$, where $X^{(2r)}$ is the $2r$-th dual of X. Then $\partial_{\mathcal{U}}^k f$ is an element of $X^{(2k)}$.

Proposition 2.3. *Let $f : G_k(\mathbb{N}) \to \mathbb{R}$ be a bounded map. Then for every $\varepsilon > 0$, there exists $\mathbb{M}$, an infinite subset of $\mathbb{N}$, such that:*

$$\forall \overline{n} \in G_k(\mathbb{M}), |f(\overline{n}) - \partial_{\mathcal{U}}^k f| < \varepsilon.$$

Proof. Let $\varepsilon > 0$. By induction on $j \in \mathbb{N}$, we will construct $\mathbb{M} = \{m_1, \ldots, m_j, \ldots\}$ such that if $\overline{n} \subset \{m_1, \ldots, m_j\}$ is of size $i \le \min\{j, k\}$, then $|\partial_{\mathcal{U}}^{k-i} f(\overline{n}) - \partial_{\mathcal{U}}^k f| < \varepsilon$:

- Because
$$\partial_{\mathcal{U}}^k f = w^* \text{ - } \lim_{n_1 \in \mathcal{U}} \cdots \lim_{n_k \in \mathcal{U}} f(n_1, \ldots, n_k)$$
and for $m \in \mathbb{N}$,
$$\partial_{\mathcal{U}}^{k-1} f(m) = w^* \text{ - } \lim_{n_2 \in \mathcal{U}} \cdots \lim_{n_k \in \mathcal{U}} f(m, n_2, \ldots, n_k)$$
we can deduce that there exists $m_1 \in \mathbb{N}$ such that $|\partial_{\mathcal{U}}^{k-1} f(m_1) - \partial_{\mathcal{U}}^k f| < \varepsilon$.
- Assume $m_1 < \cdots < m_j$ chosen.
Let $1 \le i \le \min\{j, k-1\}$ and $\overline{n} = \{n_1, \ldots, n_i\} \subset \{m_1, \ldots, m_j\}$. Then for $m > m_j$,

$$|\partial_{\mathcal{U}}^{k-(i+1)} f(\overline{n} \cup m) - \partial_{\mathcal{U}}^{k-i} f(\overline{n})| \le w^*\text{-} \lim_{n_{i+1} \in \mathcal{U}} \lim_{n_{i+2} \in \mathcal{U}} \cdots \lim_{n_k \in \mathcal{U}} |f(n_1, \ldots, n_i, m, n_{i+2}, \ldots, n_k) - f(n_1, \ldots, n_i, n_{i+1}, n_{i+2}, \ldots, n_k)|$$

Thus there exists $\mathbb{A}_{\overline{n}} \in \mathcal{U}$ such that for every $m \in \mathbb{A}_{\overline{n}}$, $m > m_j$ and

$$w^*\text{-} \lim_{n_{i+1} \in \mathcal{U}} \left(\lim_{n_{i+2} \in \mathcal{U}} \cdots \lim_{n_k \in \mathcal{U}} |f(n_1, \ldots, n_i, m, n_{i+2}, \ldots, n_k) - f(n_1, \ldots, n_i, n_{i+1}, n_{i+2}, \ldots, n_k)| \right) < \varepsilon$$

Moreover the intersection $\mathbb{A}$ of all $\mathbb{A}_{\overline{n}}$ is not empty and belongs to $\mathcal{U}$. Thus pick $m_{j+1} \in \mathbb{A}$.

Then for every $\overline{n} = \{n_1, \ldots, n_i\} \subset \{m_1, \ldots, m_j\}, 1 \le i \le \min\{j, k-1\}$,

$$|\partial_{\mathcal{U}}^{k-(i+1)} f(\overline{n} \cup m_{j+1}) - \partial_{\mathcal{U}}^k f| \le |\partial_{\mathcal{U}}^{k-(i+1)} f(\overline{n} \cup m_{j+1}) - \partial_{\mathcal{U}^{k-i}} f(\overline{n})| + |\partial_{\mathcal{U}}^{k-i} f(\overline{n}) - \partial_{\mathcal{U}}^k f| < 2\varepsilon$$

We deduce the result with $i = k$. □

It is possible to generalize this result to bounded maps which takes values into a Banach space X.

Lemma 2.4. *Let $f : G_k(\mathbb{N}) \to X$ be a bounded map. Then for every $\varepsilon > 0$, there exists $\mathbb{M}$, an infinite subset of $\mathbb{N}$, such that:*

$$\forall \overline{n} \in G_k(\mathbb{M}), \|f(\overline{n})\| < \|\partial_{\mathcal{U}}^k f\| + \omega_f(1) + \varepsilon.$$

Proof. For two bounded maps $f : G_k(\mathbb{N}) \to X$ and $g : G_k(\mathbb{N}) \to X^*$, define $f \otimes g : G_{2k}(\mathbb{N}) \to \mathbb{R}$ by $f \otimes g(\overline{n}) = \langle f(n_2, n_4, \ldots, n_{2k}), g(n_1, n_3, \ldots, n_{2k-1})\rangle$.

Then $\partial_{\mathcal{U}}^2(f \otimes g) = \partial_{\mathcal{U}} f \otimes \partial_{\mathcal{U}} g$. Indeed,

$$\begin{aligned}\partial_{\mathcal{U}}(f \otimes g)(n_1, \ldots, n_{2k-1}) &= \lim_{n_{2k} \in \mathcal{U}} \langle f(n_2, n_4, \ldots, n_{2k}), g(n_1, n_3, \ldots, n_{2k-1})\rangle \\ &= \langle \partial_{\mathcal{U}} f(n_2, \ldots, n_{2k-2}), g(n_1, \ldots, n_{2k-1})\rangle\end{aligned}$$

thus

$$\begin{aligned}\partial_{\mathcal{U}}^2(f \otimes g)(n_1, \ldots, n_{2k-2}) &= \lim_{n_{2k-1} \in \mathcal{U}} \langle \partial_{\mathcal{U}} f(n_2, n_4, \ldots, n_{2k-2}), g(n_1, n_3, \ldots, n_{2k-1})\rangle \\ &= \langle \partial_{\mathcal{U}} f(n_2, \ldots, n_{2k-2}), \partial_{\mathcal{U}} g(n_1, \ldots, n_{2k-3})\rangle \\ &= (\partial_{\mathcal{U}} f \otimes \partial_{\mathcal{U}} g)(n_1, \ldots, n_{2k-2}).\end{aligned}$$

In particular, $\partial_{\mathcal{U}}^{2k}(f \otimes g) = \partial_{\mathcal{U}}^k f \otimes \partial_{\mathcal{U}}^k g$.

Let $f : G_k(\mathbb{N}) \to X$ be a bounded map. Hahn-Banach theorem gives a map g from $G_k(\mathbb{N})$ to X^* such that for every $\overline{n} \in G_k(\mathbb{N})$, $\langle f(\overline{n}), g(\overline{n})\rangle = \|f(\overline{n})\|$ and $\|g(\overline{n})\| = 1$. It follows,

$$|\partial_{\mathcal{U}}^{2k}(f \otimes g)| = |\partial_{\mathcal{U}}^k f \otimes \partial_{\mathcal{U}}^k g| = |\langle \partial_{\mathcal{U}}^k f, \partial_{\mathcal{U}}^k g\rangle| \leq \|\partial_{\mathcal{U}}^k f\| \|\partial_{\mathcal{U}}^k g\| = \|\partial_{\mathcal{U}}^k f\|$$

The map $f \otimes g : G_{2k}(\mathbb{N}) \to \mathbb{R}$ is bounded, then we can apply Proposition 2.3 and for every $\varepsilon > 0$ there exists $\mathbb{A}$ an infinite subset of $\mathbb{N}$ such that for every $\overline{n} \in G_{2k}(\mathbb{A}), |f \otimes g(\overline{n}) - \partial_{\mathcal{U}}^{2k} f \otimes g| < \varepsilon$, hence

$$|f \otimes g(\overline{n})| < \varepsilon + |\partial_{\mathcal{U}}^{2k} f \otimes g| \leq \varepsilon + \|\partial_{\mathcal{U}}^k f\|.$$

Now we enumerate $\mathbb{A} = \{m_1 < n_1 < m_2 < n_2 < \cdots < m_j < n_j < \ldots\}$ and set $\mathbb{M} = \{m_1, \ldots, m_j, \ldots\}$.

Let $\overline{n} \in G_k(\mathbb{M})$, then for any $\overline{p} \in G_k(\mathbb{A})$ which is adjacent to $\overline{n}$ (such a $\overline{p}$ exists by the definitions of $\mathbb{A}$ and $\mathbb{M}$), we have

$$\begin{aligned}\|f(\overline{n})\| &= \langle f(\overline{n}), g(\overline{n})\rangle = \langle f(\overline{p}), g(\overline{n})\rangle + \langle f(\overline{n}) - f(\overline{p}), g(\overline{n})\rangle \\ &\leq f \otimes g(n_1, p_1, \ldots, n_k, p_k) + \|f(\overline{n}) - f(\overline{p})\| \|g(\overline{n})\| \\ &< \varepsilon + \|\partial_{\mathcal{U}}^k f\| + \omega_f(d(\overline{n}, \overline{p})) = \varepsilon + \|\partial_{\mathcal{U}}^k f\| + \omega_f(1)\end{aligned}$$

□

We can now state the result we will use to prove the main theorem:

Corollary 2.5. *Let X be a reflexive Banach space and $f : G_k(\mathbb{N}) \to X$ be a bounded map. Then for every $\varepsilon > 0$, there exists $\mathbb{M}$, an infinite subset of $\mathbb{N}$, and $x \in X$ such that:*

$$\forall \overline{n} \in G_k(\mathbb{M}), \|f(\overline{n}) - x\| \leq \omega_f(1) + \varepsilon.$$

Proof. Since X is reflexive there exists $x \in X$ such that $\partial^k_{\mathcal{U}} f = x$. We define a bounded map $g : G_k(\mathbb{N}) \to X$ by $g(\overline{n}) = f(\overline{n}) - x$, for all $\overline{n} \in G_k(\mathbb{N})$. Clearly $\partial^k_{\mathcal{U}} g = 0$ and $\omega_g(1) = \omega_f(1)$. Finally by a direct application of the previous lemma:

$$\forall \varepsilon > 0, \exists \mathbb{M} \subseteq \mathbb{N} : \forall \overline{n} \in G_k(\mathbb{M}), \|g(\overline{n})\| < \|\partial^k_{\mathcal{U}} g\| + \omega_g(1) + \varepsilon.$$

That is,

$$\forall \varepsilon > 0, \exists \mathbb{M} \subseteq \mathbb{N} : \forall \overline{n} \in G_k(\mathbb{M}), \|f(\overline{n}) - x\| < \omega_f(1) + \varepsilon.$$

□

3. Obstruction to uniform or coarse embeddings into reflexive Banach spaces

Given (M, d) a metric space, $\varepsilon > 0$ and $\delta \geq 0$, we say that M has the $\mathcal{Q}(\varepsilon, \delta)$-property if for every $k \in \mathbb{N}$, for every map $f : G_k(\mathbb{N}) \to M$ with $\omega_f(1) \leq \delta$, there exists an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that:

$$\forall \overline{n} < \overline{m} \in G_k(\mathbb{M}),\ d(f(\overline{n}), f(\overline{m})) \leq \varepsilon.$$

We define $\Delta_M(\varepsilon)$ as the supremum over all $\delta \geq 0$ such that M has the $\mathcal{Q}(\varepsilon, \delta)$-property.

The key result of this paper is the following:

Theorem 3.1. *Let (M, d) be a metric space.*

(1) *If M uniformly embeds into a reflexive Banach space, then*

$$\forall \varepsilon > 0, \Delta_M(\varepsilon) > 0.$$

(2) *If M coarsely embeds into a reflexive Banach space, then*

$$\lim_{\varepsilon \to +\infty} \Delta_M(\varepsilon) = +\infty.$$

Proof. Let X be a reflexive Banach space and $h : M \to X$ be any map.

We will prove that for every $\delta > 0$ and $f : G_k(\mathbb{N}) \to M$ a map such that $\omega_f(1) \leq \delta$, there exists an infinite subset $\mathbb{M}$ of $\mathbb{N}$ so that for every $\overline{n} < \overline{p} \in G_k(\mathbb{M})$, $\varphi_h(d(f(\overline{n}), f(\overline{p}))) \leq 4\ \omega_h(\delta)$ and conclude.

Let $\delta > 0$ and $f : G_k(\mathbb{N}) \to M$ be a map such that $\omega_f(1) \leq \delta$. We can apply Corollary 2.5 on the map $h \circ f : G_k(\mathbb{N}) \to X$, with $\varepsilon = \omega_{h \circ f}(1)$, to obtain $\mathbb{M}$, an infinite subset of $\mathbb{N}$, and $x \in X$ such that for every $\overline{n}, \overline{p} \in G_k(\mathbb{M})$,

$$\|h \circ f(\overline{n}) - h \circ f(\overline{p})\| \leq \|h \circ f(\overline{n}) - x\| + \|h \circ f(\overline{p}) - x\| \leq 4\ \omega_{h \circ f}(1) \leq 4\ \omega_h(\delta)$$

The last inequality holds because we clearly have $\omega_{h \circ f}(1) \leq \omega_h(\delta)$.

(1) **Uniform embedding.** Let $\varepsilon > 0$, then there exists $\alpha > 0$ such that $\varphi_h(\varepsilon) \geq 4\ \alpha$ and $\delta > 0$ so that $\omega_h(\delta) \leq \alpha$.

For this $\delta > 0$, for every $f : G_k(\mathbb{N}) \to M$ such that $\omega_f(1) \leq \delta$, there exists an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that $\forall \overline{n} < \overline{p} \in G_k(\mathbb{M})$,

$$\varphi_h(d(f(\overline{n}), f(\overline{p}))) \leq 4\ \omega_h(\delta) \leq 4\ \alpha \leq \varphi_h(\varepsilon).$$

We finally conclude that $d(f(\overline{n}), f(\overline{p})) \leq \varepsilon$, M has the $\mathcal{Q}(\varepsilon, \delta)$-property and $\Delta_M(\varepsilon) > 0$.

(2) **Coarse embedding.** Let $\delta > 0$, then there exist $\beta > 0$ such that $\omega_h(\delta) \leq \beta$ and $t > 0$ such that $\varphi_h(t) \geq 4\ \beta$.

Let ε be greater than t. Then for every $f : G_k(\mathbb{N}) \to M$ such that $\omega_f(1) \leq \delta$, there exists an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that $\forall \overline{n} < \overline{p} \in G_k(\mathbb{M})$,

$$\varphi_h(d(f(\overline{n}), f(\overline{p}))) \leq 4\ \omega_h(\delta) \leq 4\ \beta \leq \varphi_h(t) \leq \varphi_h(\varepsilon).$$

Then $d(f(\overline{n}), f(\overline{p})) \leq \varepsilon$ and $\Delta_M(\varepsilon) \geq \delta$. To conclude, $\lim_{\varepsilon \to +\infty} \Delta_M(\varepsilon) = +\infty$.

Which completes the proof. □

In the case where X is a Banach space, the function Δ_X has some particular properties:

Lemma 3.2. *Let X be a Banach space.*

(1) *There exists $0 \leq \mathcal{Q}_X \leq 1$ such that for every $\varepsilon > 0$, $\Delta_X(\varepsilon) = \mathcal{Q}_X \cdot \varepsilon$.*
(2) *For every $0 < \varepsilon \leq 1$, we have $\Delta_X(\varepsilon) = \Delta_{B_X}(\varepsilon)$.*

Proof.

(1) To prove that there exists a constant $\mathcal{Q}_X \geq 0$ such that for every $\varepsilon > 0$, $\Delta_X(\varepsilon) = \mathcal{Q}_X \cdot \varepsilon$, it is enough to prove that for every $\lambda > 0$, we have $\Delta_X(\lambda \cdot \varepsilon) = \lambda \cdot \Delta_X(\varepsilon)$. To do so consider $\delta > 0$ and prove that $\delta \leq \Delta_X(\lambda \cdot \varepsilon)$ is equivalent to $\delta \leq \lambda \cdot \Delta_X(\varepsilon)$, exanching the role played by the fonctions f and f/λ.

We will now prove that $\Delta_X(1) \leq 1$ and then conclude that $\mathcal{Q}_X \leq 1$.

Consider $(x_n)_{n \in \mathbb{N}}$ a sequence in X such that for all $m \neq n$, $\|x_n - x_m\| = 1$ and $f : G_1(\mathbb{N}) \to X$ defined by $f(n) = x_n, \forall n \in \mathbb{N}$. In this case $\omega_f(1) = 1$ and for every $n \neq m$, $\|f(n) - f(m)\| = 1$, thus $\mathcal{Q}_X = \Delta_X(1) \leq 1$.

(2) Finally let $0 \leq \varepsilon \leq 1$ and prove $\Delta_{B_X}(\varepsilon) = \Delta_X(\varepsilon)$.
   - Because B_X is a subset of X it is easy to see that $\Delta_{B_X}(\varepsilon) \geq \Delta_X(\varepsilon)$ for all $\varepsilon > 0$.
   - Let $k \in \mathbb{N}$ and $f : G_k(\mathbb{N}) \to X$ be a map.
     Remark that if there exists an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that for every $\overline{n} < \overline{m} \in G_k(\mathbb{M})$, $\|f(\overline{n}) - f(\overline{m})\| \leq \varepsilon$, then the image of $G_k(\mathbb{M})$ by f belongs to a ball of radius 1. Indeed if $\mathbb{M} = \{m_1 < \cdots < m_k < \cdots\}$, denote $\overline{m} = (m_1, \ldots, m_k)$ and $\mathbb{M}' = \{m_{k+1} < \cdots < m_j < \ldots\}$. Then for every $\overline{n} \in G_k(\mathbb{M}')$, we have $\|f(\overline{n}) - f(\overline{m})\| \leq \varepsilon \leq 1$, thus $f(G_k(\mathbb{M}')) \subseteq f(\overline{m}) + B_X$.
     So we can consider only $f : G_k(\mathbb{N}) \to X$ so that there exits $\mathbb{M}$ and $x_0 \in X$ such that $f(G_k(\mathbb{M})) \subseteq x_0 + B_X$ and $\omega_f(1) \leq \Delta_{B_X}(\varepsilon)$. Now for $\overline{n} \in G_k(\mathbb{M})$ define $g(\overline{n}) = f(\overline{n}) - x_0$. Because $g : G_k(\mathbb{M}) \to B_X$ and $\omega_g(1) \leq \Delta_{B_X}(\varepsilon)$, there exists $\mathbb{M}'$ an infinite subset of $\mathbb{M}$ such that for every $\overline{n} < \overline{m} \in G_k(\mathbb{M}')$, $\|g(\overline{n}) - g(\overline{m})\| \leq \varepsilon$, that is $\|f(\overline{n}) - f(\overline{m})\| \leq \varepsilon$.
     Finally we can conclude that $\Delta_X(\varepsilon) \geq \Delta_{B_X}(\varepsilon)$.

□

Thanks to this Lemma we are ready to define the so called $\mathcal{Q}$-property:

Definition 3.3. We say that a Banach space X has the $\mathcal{Q}$-property if $\mathcal{Q}_X > 0$.

We can use Theorem 3.1 in order to give an obstruction to uniform or coarse embeddings into reflexive Banach spaces in terms of property $\mathcal{Q}$.

Corollary 3.4. *Let X be a Banach space which fails the $\mathcal{Q}$-property. Then*
(1) *B_X cannot be uniformly embedded into a reflexive Banach space.*
(2) *X cannot be coarsely embedded into a reflexive Banach space.*

Proof.
(1) Assume that B_X uniformly embeds into a reflexive Banach space. Then for every positive ε, $\Delta_{B_X}(\varepsilon) > 0$. But $\Delta_{B_X}(1) = \Delta_X(1) = \mathcal{Q}_X \cdot 1 > 0$, so finally X has the $\mathcal{Q}$-property.
(2) Assume that X coarsely embeds into a reflexive Banach space. Then $\lim\limits_{\varepsilon\to+\infty} \mathcal{Q}_X \cdot \varepsilon = \lim\limits_{\varepsilon\to+\infty} \Delta_X(\varepsilon) = +\infty$, hence $\mathcal{Q}_X \neq 0$ and X has the $\mathcal{Q}$-property.

□

4. Examples

4.1. **Reflexive spaces.** It is clear by Corollary 3.4 that a reflexive Banach space has the $\mathcal{Q}$-property.

4.2. **Stable spaces.** Recall that a metric space (M, d) is stable if for every sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ in M, if the following limits exist, then

$$\lim_{m\to+\infty}\lim_{n\to+\infty} d(x_m, y_n) = \lim_{n\to+\infty}\lim_{m\to+\infty} d(x_m, y_n).$$

It is proved in Section 2 of [11] that a stable metric space strongly uniformly embeds into a reflexive Banach space. So we deduce that a stable Banach space has the $\mathcal{Q}$-property. But we will prove this by another way: the next proposition is proved by a Ramsey type argument.

Proposition 4.1. *Let (M, d) be a stable metric space and $f : G_k(\mathbb{N}) \to M$ a bounded map. Then for every $\varepsilon > 0$ there exists $\mathbb{M}$, an infinite subset of $\mathbb{N}$, such that for every $\overline{n} < \overline{m} \in G_k(\mathbb{M})$,*

$$d(f(\overline{n}), f(\overline{m})) < \omega_f(1) + \varepsilon.$$

Proof. Since f is bounded, applying Theorem 2.1, we can find an infinite subset $\mathbb{M}$ of $\mathbb{N}$ and $a > 0$ such that for every $\overline{p}, \overline{q} \in G_k(\mathbb{M})$, $|d(f(\overline{p}), f(\overline{q})) - a| < \frac{\varepsilon}{4}$.

Let $\mathcal{U}$ be a non-principal ultrafilter which contains $\mathbb{M}$. Then,

$$\lim_{m_1\in\mathcal{U}}\lim_{n_1\in\mathcal{U}}\dots\lim_{m_k\in\mathcal{U}}\lim_{n_k\in\mathcal{U}} d(f(\overline{n}), f(\overline{m})) \leq \omega_f(1)$$

and because M is stable (see Lemma 9.19 in [6]),

$$\lim_{m_1\in\mathcal{U}}\dots\lim_{m_k\in\mathcal{U}}\lim_{n_1\in\mathcal{U}}\dots\lim_{n_k\in\mathcal{U}} d(f(\overline{n}), f(\overline{m})) \leq \omega_f(1).$$

Then, one can find $m_1 \leq \dots \leq m_k \leq n_1 \leq \dots \leq n_k$ such that

$$d(f(\overline{n}), f(\overline{m})) < \omega_f(1) + \frac{\varepsilon}{4}.$$

Therefore,

$$a < d(f(\overline{n}), f(\overline{m})) + \frac{\varepsilon}{4} < \omega_f(1) + \frac{\varepsilon}{2}.$$

Finally for every $\overline{p}, \overline{q} \in G_k(\mathbb{M})$,

$$d(f(\overline{p}), f(\overline{q})) < \frac{\varepsilon}{4} + a < \omega_f(1) + \varepsilon.$$

□

Corollary 4.2. *A stable Banach space X has the $\mathcal{Q}$-property.*

Proof. Let $\varepsilon > 0$ and $f : G_k(\mathbb{N}) \to X$ be such that $\omega_f(1) \leq \frac{\varepsilon}{2}$. In particular f is bounded and we can use the previous proposition to obtain an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that for every $\overline{n} < \overline{m} \in G_k(\mathbb{M})$, $\|f(\overline{n}) - f(\overline{m})\| \leq \omega_f(1) + \frac{\varepsilon}{2} \leq \varepsilon$, that is X has the $\mathcal{Q}$-property. □

4.3. **Some Banach spaces failing the $\mathcal{Q}$-property.** The following result will be useful to prove that some spaces do not have the $\mathcal{Q}$-property.

Theorem 4.3. *Let X be a Banach space with the $\mathcal{Q}$-property. Then for every $\varepsilon > 0$ and every $(x_n)_{n\in\mathbb{N}}$ bounded sequence in X with a w^*-cluster point $x^{**} \in X^{**}$, there exists a subsequence $(y_n)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that*

$$\forall k \in \mathbb{N},\ \forall \overline{n} \in G_{2k}(\mathbb{N}),\ \|\sum_{j=1}^{2k}(-1)^j y_{n_j}\| \geq (1-\varepsilon)\mathcal{Q}_X k d(x^{**}, X).$$

Proof. Let $\varepsilon > 0$ and $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in X with a w^*-cluster point $x^{**} \in X^{**}$. We will denote $B := \sup_{n\in\mathbb{N}} \|x_n\|$ and $\theta := d(x^{**}, X)$. We can assume that $\theta > 0$. Let $\lambda > 1$ and $P \in \mathbb{N}$ be such that $\frac{1}{\lambda^2} \geq 1 - \frac{\varepsilon}{2}$ and $\frac{1}{P} \leq \frac{\varepsilon \mathcal{Q}_X \theta}{2(2B+\theta)}$.

First it is possible to extract a subsequence $(v_n)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that for every $1 \leq m < n$ and every sequence $(a_j)_{j=1}^n$ of positive numbers such that

$$\sum_{j=1}^m a_j = \sum_{j=m+1}^n a_j = 1,$$

we have

$$\|\sum_{j=1}^m a_j v_j - \sum_{j=m+1}^n a_j v_j\| > \frac{\theta}{\lambda}.$$

We will prove that one can find a subsequence $(y_n)_{n\in\mathbb{N}}$ of $(v_n)_{n\in\mathbb{N}}$ such that for every $k \geq 1$, there exists $b_k > 0$ such that for every $\overline{n} \in G_{2k}(\mathbb{N})$,

$$b_k - \frac{\theta}{P} \leq \|\sum_{j=1}^{2k}(-1)^j y_{n_j}\| \leq b_k.$$

Consider first $g_1 : \begin{array}{ccc} G_2(\mathbb{N}) & \to & \mathbb{R} \\ \overline{n} & \mapsto & \|v_{n_1} - v_{n_2}\| \end{array}$. Since the sequence $(v_n)_{n\in\mathbb{N}}$ is bounded, using Ramsey's theorem, one can find $b_1 > 0$ and $\varphi_1 : \mathbb{N} \to \mathbb{N}$ an increasing bijection such that:

$$\forall \overline{n} \in G_2(\varphi_1(\mathbb{N})),\ b_1 - \frac{\theta}{P} \leq g_1(\overline{n}) \leq b_1.$$

Now for a fixed $k \in \mathbb{N}$, assume that for every $1 \leq l \leq k-1$, φ_l is constructed such that $\varphi_l(\mathbb{N})$ is extracted from $\varphi_{l-1}(\mathbb{N})$. Consider $g_k : \begin{array}{ccc} G_{2k}(\mathbb{N}) & \to & \mathbb{R} \\ \overline{n} & \mapsto & \|\sum_{j=1}^{2k}(-1)^j v_{n_j}\| \end{array}$.

As previously there exists $b_k > 0$ and $\varphi_k : \mathbb{N} \to \mathbb{N}$ such that:

$$\forall \overline{n} \in G_{2k}(\varphi_1 \circ \cdots \circ \varphi_k(\mathbb{N})),\ b_k - \frac{\theta}{P} \leq g_k(\overline{n}) \leq b_k.$$

If we define $\psi : \begin{array}{ccc} \mathbb{N} & \to & \mathbb{N} \\ n & \mapsto & \varphi_1 \circ \cdots \circ \varphi_{P \cdot n}(n) \end{array}$, we obtain that if $n_1 \geq \frac{k}{P}$, then $\psi(n_1) = \varphi_1 \circ \cdots \circ \varphi_k(n_1)$. Thus the subsequence $(v_{\psi(n)})_{n\in\mathbb{N}}$ verifies: for every $k \in \mathbb{N}$, there exists a constant b_k such that $\forall \overline{n} \in G_{2k}(\mathbb{N})$ verifying $n_1 \geq \frac{k}{P}$, we have

$$b_k - \frac{\theta}{P} \leq \| \sum_{j=1}^{2k} (-1)^j v_{\psi(n_j)} \| \leq b_k.$$

We will denote the subsequence $(v_{\psi(n)})_{n\in\mathbb{N}}$ by $(y_n)_{n\in\mathbb{N}}$.

Fix $k \in \mathbb{N}$ and set $\mathbb{M} = \{n \in \mathbb{N};\ n \geq \frac{k}{P}\}$. Define $f : \begin{array}{ccc} G_k(\mathbb{M}) & \to & X \\ \overline{n} & \mapsto & \sum_{j=1}^{k} y_{n_j} \end{array}$. We have

$$\omega_f(1) = \sup \left\{ \| \sum_{j=1}^{k} y_{n_j} - \sum_{j=1}^{k} y_{m_j} \|;\ \frac{k}{P} \leq m_1 < n_1 < m_2 < \cdots < m_k < n_k \right\}$$
$$= \sup \left\{ \| \sum_{j=1}^{2k} (-1)^j y_{n_j} \|;\ \frac{k}{P} \leq n_1 < \cdots < n_{2k} \right\} \leq b_k.$$

Since X has the $\mathcal{Q}$-property, there exists $\mathbb{M}'$, an infinite subset of $\mathbb{M}$, such that for every $\overline{n} < \overline{m} \in G_k(\mathbb{M}')$, $\|f(\overline{n}) - f(\overline{m})\| \leq \dfrac{b_k}{\mathcal{Q}_X}$. So,

$$k \cdot \frac{\theta}{\lambda} < k \cdot \| \sum_{j=1}^{k} \frac{1}{k} y_{n_j} - \sum_{j=1}^{k} \frac{1}{k} y_{m_j} \| = \|f(\overline{n}) - f(\overline{m})\| \leq \frac{b_k}{\mathcal{Q}_X} < \frac{b_k}{\mathcal{Q}_X} \cdot \lambda$$

that is $b_k \geq \dfrac{\mathcal{Q}_X \cdot k \cdot \theta}{\lambda^2}$.

Now if $\overline{n} \in G_{2k}(\mathbb{N})$, one can find $\overline{m} \in G_{2k}(\mathbb{M})$ such that

$$\| \sum_{j=1}^{2k} (-1)^j y_{n_j} + \sum_{j=1}^{2k} (-1)^j y_{m_j} \| \leq \frac{2Bk}{P} \quad \text{or} \quad \| \sum_{j=1}^{2k} (-1)^j y_{n_j} - \sum_{j=1}^{2k} (-1)^j y_{m_j} \| \leq \frac{2Bk}{P}.$$

Finally,

$$\| \sum_{j=1}^{2k} (-1)^j y_{n_j} \| \geq \| \sum_{j=1}^{2k} (-1)^j y_{m_j} \| - \frac{2Bk}{P} \geq b_k - \frac{\theta}{P} - \frac{2Bk}{P} > b_k - \frac{k\theta}{P} - \frac{2Bk}{P}$$
$$\geq \frac{\mathcal{Q}_X k\theta}{\lambda^2} - \left(\frac{\varepsilon \mathcal{Q}_X k\theta}{2(2B+\theta)} \right) (\theta + 2B) \geq k\theta \mathcal{Q}_X (1 - \varepsilon),$$

which concludes the proof. □

Corollary 4.4. *The James space J and its dual J^* fail the $\mathcal{Q}$-property. In particular they cannot be uniformly or coarsely embedded into a reflexive Banach space.*

Proof. Let $(e_n)_{n\in\mathbb{N}}$ be the canonical basis of J and $x_n = \sum_{j=1}^{n} e_j$, $n \in \mathbb{N}$. With the notations of Theorem 4.3, we have $x^{**} = (1, \dots, 1, \dots) \in J^{**}$ and $d(x^{**}, X) = 1$. For every $k \in \mathbb{N}$ and every $\overline{n} \in G_{2k}(\mathbb{N})$,

$$\|\sum_{j=1}^{2k}(-1)^j x_{n_j}\|_J = (2k)^{1/2}$$

Finally assume J has the $\mathcal{Q}$-property, that is $\mathcal{Q}_J > 0$. Then for every $\varepsilon \leq 1$, one can find $k \in \mathbb{N}$ such that $(1-\varepsilon)\mathcal{Q}_J k \geq (2k)^{1/2}$. Thus $(x_n)_{n\in\mathbb{N}}$ does not verify the conclusion of Theorem 4.3.

In the case of J^* we consider the sequence $(e^*_n)_{n\in\mathbb{N}}$ which converges to an element of J^{***} of norm 1. Moreover for every $k \in \mathbb{N}$ and every $\overline{n} \in G_{2k}(\mathbb{N})$, we have $\|\sum_{j=1}^{2k}(-1)^j e^*_{n_j}\|_{J^*} \leq k^{1/2}$ and we conclude as previously.

The second part of the result is just a consequence of Corollary 3.4. □

4.4. **The space c_0.**

Corollary 4.5. *The space c_0 fails the $\mathcal{Q}$-property. In particular c_0 cannot be uniformly or coarsely embedded into a reflexive Banach space.*

Proof. We will prove that the summing bases of c_0 does not verify the conclusion of Theorem 4.3.

Let $(e_n)_{n\in\mathbb{N}}$ be the canonical bases of c_0 and $x_n = \sum_{j=1}^{n} e_j$, $n \in \mathbb{N}$. With the notation of Theorem 4.3, we have $x^{**} = (1, \dots, 1, \dots)$ and $d(x^{**}, X) = 1$. It is clear that for every $k \in \mathbb{N}$ and every $\overline{n} \in G_{2k}(\mathbb{N})$, we have $\|\sum_{j=1}^{2k}(-1)^j x_{n_j}\| = 1$.

Finally assume c_0 has the $\mathcal{Q}$-property, that is $\mathcal{Q}_{c_0} > 0$. Then for every $\varepsilon \leq 1$, one can find $k \in \mathbb{N}$ such that $(1-\varepsilon)\mathcal{Q}_{c_0} k \geq 1$. Thus $(x_n)_{n\in\mathbb{N}}$ does not verify the conclusion of Theorem 4.3.

The second part of the result is a consequence of Corollary 3.4. □

In fact in [11] Kalton proved, before the introduction of the $\mathcal{Q}$-property, that c_0 cannot be uniformly or coarsely embedded into a Banach space such that all its iterated duals are separable. This result is stronger because all iterated duals of J are separable and this space fails the $\mathcal{Q}$-property.

Theorem 4.6. *Let X be a Banach space such that all its duals are separable. Then c_0 cannot be uniformly or coarsely embedded into X.*

Lemma 4.7. *Let X be a Banach space such that for every $k \in \mathbb{N}$, the $2k$-th dual $X^{(2k)}$ of X is separable. Then for every uncountable family $(f_i)_{i\in I}$ of bounded functions $f_i : G_k(\mathbb{N}) \to X$ and for every $\varepsilon > 0$, there exist $i \neq j$ and $\mathbb{M}$, an infinite subset of $\mathbb{N}$, such that*

$$\forall \overline{n} \in G_k(\mathbb{M}), \|f_i(\overline{n}) - f_j(\overline{n})\| < \omega_{f_i}(1) + \omega_{f_j}(1) + \varepsilon.$$

Proof. For every $i \in I$, $\partial^k_{\mathcal{U}} f_i$ belongs to $X^{(2k)}$ and this space is separable thus there exist $i \neq j$ such that $\|\partial^k_{\mathcal{U}} f_i - \partial^k_{\mathcal{U}} f_j\| < \frac{\varepsilon}{2}$.

Now if we apply Lemma 2.4 to $f_i - f_j$, we obtain $\mathbb{M}$ an infinite subset of $\mathbb{N}$ such that for every $\overline{n} \in G_k(\mathbb{M})$,

$$\begin{aligned}\|f_i(\overline{n}) - f_j(\overline{n})\| = \|(f_i - f_j)(\overline{n})\| &< \|\partial^k_{\mathcal{U}}(f_i - f_j)\| + \omega_{f_i - f_j}(1) + \frac{\varepsilon}{2} \\ &< \omega_{f_i - f_j}(1) + \varepsilon \leq \omega_{f_i}(1) + \omega_{f_j}(1) + \varepsilon.\end{aligned}$$

□

Proof of Theorem 4.6: Let X be a Banach space having all its iterated duals separable and $h : c_0 \to X$ be a map. We will prove that h cannot be a coarse or uniform embedding. First, we can assume that h is bounded on bounded sets.

Let $(e_n)_{n\in\mathbb{N}}$ be the canonical basis of c_0 and define, for every $\mathbb{A}$ infinite subset of $\mathbb{N}$,

$$s_{\mathbb{A}}(n) = \sum_{\substack{r \leq n \\ r \in \mathbb{A}}} e_r, n \in \mathbb{N}.$$

Let $k \in \mathbb{N}$ and $0 < t < +\infty$ and define, for every $\mathbb{A}$ infinite subset of $\mathbb{N}$,

$$f_{\mathbb{A}} : \begin{array}{ccc} G_k(\mathbb{N}) & \to & c_0 \\ \overline{n} & \mapsto & t \sum\limits_{j=1}^{k} s_{\mathbb{A}}(n_j) \end{array}$$

We have $\{h \circ f_{\mathbb{A}};\ \mathbb{A}$ infinite subset of $\mathbb{N}\}$ an uncountable family of bounded functions $h \circ f_{\mathbb{A}} : G_k(\mathbb{N}) \to X$, then we can apply Lemma 4.7: for every $\varepsilon > 0$, there exist $\mathbb{A} \neq \mathbb{B}$ and $\mathbb{M}$, infinite subsets of $\mathbb{N}$, such that

$$\forall \overline{n} \in G_k(\mathbb{M}), \|h \circ f_{\mathbb{A}}(\overline{n}) - h \circ f_{\mathbb{B}}(\overline{n})\| < \omega_{h\circ f_{\mathbb{A}}}(1) + \omega_{h\circ f_{\mathbb{B}}}(1) + \varepsilon.$$

Moreover, we have $\omega_{h\circ f_{\mathbb{D}}}(1) \leq \omega_h(t)$, for every $\mathbb{D}$ infinite subset of $\mathbb{N}$. Indeed $\omega_{f_{\mathbb{D}}}(1) \leq t$ and $\omega_{h\circ f_{\mathbb{D}}}(1) = \omega_h(\omega_{f_{\mathbb{D}}}(1)) \leq \omega_h(t)$.

Thus we have $\mathbb{A} \neq \mathbb{B}$ and $\mathbb{M}$, infinite subsets of $\mathbb{N}$, such that for every $\overline{n} \in G_k(\mathbb{N})$,

$$\|h \circ f_{\mathbb{A}}(\overline{n}) - h \circ f_{\mathbb{B}}(\overline{n})\| < 2\omega_h(t) + \varepsilon.$$

Since $\mathbb{A} \neq \mathbb{B}$ are infinite, there exists $\overline{p} \in G_k(\mathbb{M})$ such that $\|f_{\mathbb{A}}(\overline{p}) - f_{\mathbb{B}}(\overline{p})\| = kt$. Hence, $\varphi_h(kt) \leq \|h \circ f_{\mathbb{A}}(\overline{p}) - h \circ f_{\mathbb{B}}(\overline{p})\| < 2\omega_h(t) + \varepsilon$, for every $\varepsilon > 0$. Finally we have

$$\forall k \in \mathbb{N}, \forall t > 0, \varphi_h(kt) < 2\omega_h(t).$$

We will now distinguish two cases to prove that h cannot be a coarse or a uniform embedding:

- **Uniform embedding.** If $\lim\limits_{t\to 0} \omega_h(t) = 0$, we deduce that for every $t > 0$, $\varphi_h(t) = 0$ and conclude that h cannot be a uniform embedding.
- **Coarse embedding.** If for every $t > 0$, $\omega_h(t)$ is finite, we can deduce that $\lim\limits_{t\to+\infty} \varphi_h(t)$ is finite, that is h is not a coarse embedding.

□

5. Lipschitz and uniform embeddings into ℓ_∞

To conclude we mention that in [12] Kalton follows the same ideas to prove that $\mathcal{C}[1,\omega_1]$ cannot be uniformly embedded into ℓ_∞, where ω_1 is the first uncountable ordinal.

For every $k \in \mathbb{N}$ we define $G_k(\omega_1)$ the set of all subsets of ω_1 of size k. We keep the same notations as previously and define a distance d over $G_k(\omega_1)$ in the same way. Kalton proved the following results:

Theorem 5.1 (To compare to Corollary 2.5)**.** *Let $f : G_k(\omega_1) \to \ell_\infty$ be a Lipschitz mapping with Lipschitz constant L. Then there exist $x \in \ell_\infty$ and $\Omega \subset \omega_1$ such that for every $\overline{\alpha} \in G_k(\Omega)$,*

$$\|f(\overline{\alpha}) - x\| \leq \frac{L}{2}.$$

As a corollary (to compare to Corollary 4.5) it is proved:

Corollary 5.2. *The Banach space and $\mathcal{C}[1,\omega_1]$ cannot be uniformly embedded into ℓ_∞.*

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