FREE SPACES OVER COUNTABLE COMPACT METRIC SPACES

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Abstract. We prove that the Lipschitz-free space over a countable compact metric space is isometric to a dual space and has the metric approximation property.

1. Introduction

Let (M, d) be a pointed metric space, that is to say a metric space equipped with a distinguished origin, denoted 0. The space $Lip_0(M)$ of Lipschitz functions from M to $\mathbb{R}$ vanishing at 0 is a Banach space equipped with the Lipschitz norm:

$$\|f\|_L = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)}.$$

Its unit ball is compact with respect to the pointwise topology, then $Lip_0(M)$ is a dual space. In [3], its predual is called the Lipschitz free space over M, denoted $\mathcal{F}(M)$ and it is the closed linear span of $\{\delta_x \ , \ x \in M\}$ in $Lip_0(M)^*$. One can prove that the map $\delta : M \to \mathcal{F}(M)$ is an isometry. For more details on the basic theory of the spaces of Lipschitz functions and their preduals, called Arens-Eells space there, see [14].

Very little is known about the structure of Lipschitz-free spaces. For instance $\mathcal{F}(\mathbb{R})$ is isomorphically isometric to L_1, but A. Naor and G. Schechtman [11] proved that $\mathcal{F}(\mathbb{R}^2)$ is not isomorphic to any subspace of L_1. The study of the Lipschitz-free space over a Banach space is useful to learn more about the structure of this Banach space. For example G. Godefroy and N. Kalton [3] proved, using this theory, that if a separable Banach space X isometrically embeds in a Banach space Y, then Y contains a linear subspace which is linearly isometric to X.

We recall that a Banach space X is said to have the approximation property (AP) if for every $\varepsilon > 0$ and every compact set $K \subset X$, there is a bounded finite-rank linear operator $T : X \to X$ such that $\|Tx - x\| \leq \varepsilon$ for every $x \in K$. If moreover there exists $1 \leq \lambda < +\infty$ not depending on ε or K such that $\|T\| \leq \lambda$ then X has the λ-bounded approximation property (λ-BAP) and X has the bounded approximation property (BAP) if it has the λ-BAP for some λ. Finally X has the metric approximation property (MAP) if $\lambda = 1$.

It is already known that $\mathcal{F}(\mathbb{R}^n)$ has the MAP [3], and that if M is a doubling metric space then $\mathcal{F}(M)$ has the BAP [9]. Moreover E. Pernecká and P. Hájek [7]

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proved that $\mathcal{F}(\ell_1)$ and $\mathcal{F}(\mathbb{R}^n)$ have a Schauder basis. However, G. Godefroy and N. Ozawa [4] constructed a compact metric space K such that $\mathcal{F}(K)$ fails the AP.

In the first part of this article we will prove that the Lipschitz-free space over a countable compact metric space K is isometrically isomorphic to the dual space of $lip_0(K) \subset Lip_0(K)$. Let ω_1 be the first uncountable ordinal, we will prove, by induction on $\alpha < \omega_1$ such that $K^{(\alpha)}$ is finite, that $\mathcal{F}(K)$ has the MAP. This will rely on a theorem of A. Grothendieck [6] asserting that any separable dual having the BAP has the MAP, and a decomposition of the space K due to N. Kalton [8]. This provides a negative answer to Question 2 in [4], which was originally asked by G. Aubrun to G. Godefroy during a seminar in Lyon about his paper with N. Ozawa.

2. Duality

For any pointed metric space (M, d) we denote by $lip_0(M)$ the subspace of $Lip_0(M)$ defined as follows: $f \in lip_0(M)$ if and only if for every $\varepsilon > 0$, there is a $\delta > 0$ such that for $x, y \in M$, $d(x, y) < \delta$ implies $|f(x) - f(y)| \leq \varepsilon d(x, y)$.

The main result of this section is the following:

Theorem 2.1. *If (K, d) is a countable compact metric space, then $\mathcal{F}(K)$ is isometrically isomorphic to a dual space, namely $lip_0(K)^*$.*

Definition 2.2.

(1) Let X be a Banach space. A subspace S of X^* is called separating if $x^*(x) = 0$ for all $x^* \in S$ implies $x = 0$.

(2) For (M, d) a pointed metric space, $lip_0(M)$ separates points uniformly if there exists a constant $c \geq 1$ such that for every $x, y \in M$, some $f \in lip_0(M)$ satisfies $\|f\|_L \leq c$ and $|f(x) - f(y)| = d(x, y)$.

Mimicking an argument from [2] we will use a theorem due to Petunīn and Plīčhko [13] saying that if $(X, \|\cdot\|)$ is a separable Banach space and S a closed subspace of X^* contained in $NA(X)$ (the subset of X^* consisting of all linear forms which attain their norm) and separating points of X, then X is isometrically isomorphic to S^*. Theorem 3.3.3 in [14] gives the same result but in a less general case.

We start with two lemmas taken from [2].

Lemma 2.3. *For any (K, d) compact pointed metric space, the space $lip_0(K)$ is a subset of $NA(\mathcal{F}(K))$.*

Proof. We can see $lip_0(K)$ as the subset of $Lip_0(K)$ containing all f such that for every $\varepsilon > 0$, the set $K^2_\varepsilon := \big\{(x, y) \in K^2,\ x \neq y,\ |f(x) - f(y)| \geq \varepsilon\ d(x, y)\big\}$ is compact.

Let $f \in lip_0(K)$, we may assume that $f \neq 0$, then there exists $\varepsilon > 0$ such that,

$$\|f\|_L = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)} = \sup_{(x,y) \in K^2_\varepsilon} \frac{|f(x) - f(y)|}{d(x, y)} = \max_{(x,y) \in K^2_\varepsilon} \frac{|f(x) - f(y)|}{d(x, y)}$$

Thus there exist $x \neq y$ such that $\|f\|_L = \frac{|f(x)-f(y)|}{d(x,y)}$ and setting $\gamma = \frac{1}{d(x,y)}(\delta_x - \delta_y)$ we obtain $\gamma \in \mathcal{F}(K)$ and $\|f\|_L = |f(\gamma)|$, with $\|\gamma\|_{\mathcal{F}(K)} = 1$ because δ is an isometry. Then f is norm attaining and $lip_0(K) \subset NA(\mathcal{F}(K))$. □

Lemma 2.4. *For any (K,d) compact pointed metric space, if $lip_0(K)$ separates points uniformly, then it is separating.*

Proof. Using Hahn-Banach theorem, one can prove that $lip_0(K)$ is separating if and only if it is weak*- dense in $Lip_0(K)$.

Now assume $lip_0(K)$ separates points uniformly. Then there exists $c \geq 1$ such that for every $F \subset K$, F finite, and every $f \in Lip_0(K)$, we can find $g \in lip_0(K)$, $\|g\|_L \leq c\|f\|_L$, such that $f_{|F} = g_{|F}$ (see Lemma 3.2.3 in [14]), and it is easy to deduce that $\overline{lip_0(K)}^{\,w^*} = Lip_0(K)$.

□

These lemmas allow us to reduce the problem. We need to prove that the little Lipschitz space over a countable compact metric space separates points uniformly.

For this proof we will use a characterization of countable compact metric spaces with the Cantor-Bendixon derivation: for a metric space (M,d) we denote

- M' the set of accumulation points of M.
- $M^{(\alpha)} = (M^{(\alpha-1)})'$, for a successor ordinal α.
- $M^{(\alpha)} = \bigcap\limits_{\beta<\alpha} M^{(\beta)}$, for a limit ordinal α.

A compact metric space (K,d) is countable if and only if there is a countable ordinal α such that $K^{(\alpha)}$ is finite.

Proof of Theorem 2.1: Let us prove that

$$\exists c \geq 1,\ \forall x,y \in K,\ \exists h \in lip_0(K),\ \|h\|_L \leq c,\ |h(x)-h(y)| = d(x,y).$$

So let $x \neq y \in K$ and set $a = d(x,y)$. Since K is countable and compact the closed ball $\overline{B}\left(x,\frac{a}{2}\right)$ of center x and radius $\frac{a}{2}$ is countable and compact and there exists a countable ordinal α_0 such that $\overline{B}\left(x,\frac{a}{2}\right)^{(\alpha_0)}$ is finite and non empty: there exist $k_1 \in \mathbb{N}$, $y_1^1, \cdots, y_1^{k_1} \in K$ such that $\overline{B}\left(x,\frac{a}{2}\right)^{(\alpha_0)} = \{y_1^1, \cdots, y_1^{k_1}\}$. We denote $a_1^i = d(y_1^i, x)$, for $1 \leq i \leq k_1$. Then we can find r_1 and $v_1^1 < \cdots < v_1^{r_1}$ such that $\{a_1^1, \cdots, a_1^{k_1}\} = \{v_1^1, \cdots, v_1^{r_1}\}$. Now set

$$v_1 = \begin{cases} a/2, & \text{if } \overline{B}\left(x,\frac{a}{2}\right)^{(\alpha_0)} = \{x\} \\ \min\left(\left(\{v_1^1,\ \frac{a}{2} - v_1^{r_1}\}\backslash\{0\}\right) \cup \{v_1^i - v_1^{i-1},\ 2 \leq i \leq r_1\}\right), & \text{otherwise} \end{cases}$$

and define $\varphi_1 : [0,+\infty[\to [0,+\infty[$ by

$$\varphi_1(t) = \begin{cases} 0 & ,\ t \in \left[0, \frac{v_1}{4}\right[:= V_1^0 \\ v_1^i & ,\ t \in \left]v_1^i - \frac{v_1}{4}, v_1^i + \frac{v_1}{4}\right[:= V_1^i,\ 1 \leq i \leq r_1 \\ \frac{a}{2} & ,\ t \in \left]\frac{a}{2} - \frac{v_1}{4}, +\infty\right[:= V_1^{r_1+1} \end{cases}$$

and φ_1 is continuous on $[0,+\infty[$ and affine on each interval of $[0,+\infty[\setminus \cup_{i=0}^{r_1+1} V_1^i$.
One can check that the slope of φ_1 is at most 2 on each of these intervals, so $\|\varphi_1\|_L \leq 2$.

With $f(\cdot) = d(\ \cdot\ ,x)$ we set $C_1 = f^{-1}\left([0,+\infty[\setminus \cup_{i=0}^{r_1+1} V_1^i\right)$.
If C_1 is finite or empty define $h(\cdot) = 2\left(\varphi_1 \circ d(\ \cdot\ ,x) - \varphi_1(d(0,x))\right)$. It is clear from the definition of φ_1 that $\|h\|_L \leq 4$, $|h(x)-h(y)| = d(x,y)$ and $h(0) = 0$. Now

we set

$$\delta = \begin{cases} v_1/2, & \text{if } C_1 = \emptyset \\ \dfrac{1}{2}\min\left(\{v_1, \operatorname{sep}(C_1)\} \cup \{\operatorname{dist}(z, K\backslash C_1),\ z \in D_1\}\right), & \text{otherwise} \end{cases}$$

where $\operatorname{sep}(C_1) = \inf\{d(z,t),\ z \neq t,\ z,t \in C_1\}$ and $D_1 = f^{-1}\left([0,+\infty[\backslash \cup_{i=0}^{r_1+1} \overline{V_1^i}\right)$. Note that $\delta > 0$. Indeed $v_1 > 0$, C_1 is finite thus $\operatorname{sep}(C_1) > 0$, for any $z \in D_1$ $\operatorname{dist}(z, K\backslash C_1) > 0$ and D_1 is finite.

If follows that every $z \neq t \in K$ such that $d(z,t) \leq \delta$ are not in D_1 and there exists $i \leq r_1$ such that $z, t \in f^{-1}\left(\overline{V_1^i}\right)$, so the equality $h(z) = h(t)$ holds, i.e. $h \in lip_0(K)$.

Assume that C_1 is infinite. Since $C_1 \subset \overline{B}\left(x, \frac{a}{2}\right)$ we have that for every ordinal α, $C_1^{(\alpha)} \subset \overline{B}\left(x, \frac{a}{2}\right)^{(\alpha)}$. But $C_1 \cap \overline{B}\left(x, \frac{a}{2}\right)^{(\alpha_0)} = \emptyset$ so $C_1^{(\alpha_0)} = \emptyset$. However C_1 is compact, thus there exists $1 \leq \alpha_1 < \alpha_0$ so that $C_1^{(\alpha_1)}$ is finite and non empty. Then there exist $k_2 \in \mathbb{N}$ and $y_2^1, \cdots, y_2^{k_2} \in K$, such that $C_1^{(\alpha_1)} = \{y_2^1, \cdots, y_2^{k_2}\}$.

For $1 \leq i \leq k_2$, we denote $a_2^i = d(y_2^i, x)$. We can find r_2 and $v_2^1 < \cdots < v_2^{r_2}$ such that

$$\{a_2^1, \cdots, a_2^{k_2}\} = \{v_2^1, \cdots, v_2^{r_2}\}.$$

Now set

$$v_2 = \min\left(\{v_1,\ v_2^1\} \cup \{v_2^i - v_2^{i-1},\ 2 \leq i \leq r_2\}\right)$$

and define $\varphi_2 : [0,+\infty[\to [0,+\infty[$ continuous by

$$\varphi_2(t) = \begin{cases} \varphi_1(t) & ,\ t \in \displaystyle\bigcup_{i=0}^{r_1+1} V_1^i \\ \varphi_1(v_2^i) & ,\ t \in \left]v_2^i - \frac{v_2}{2^3}, v_2^i + \frac{v_2}{2^3}\right[:= V_2^i,\ 1 \leq i \leq r_2 \end{cases}$$

and φ_2 is affine on each interval of $[0,+\infty[\backslash\left(\left(\cup_{i=0}^{r_1+1} V_1^i\right) \cup \left(\cup_{i=1}^{r_2} V_2^i\right)\right)$.

The Lipschitz constant of φ_2 equals the maximum between $\|\varphi_1\|_L$ and new slopes of φ_2. It is easy to check that $\|\varphi_2\|_L \leq 2 \times (1 + \frac{1}{3}) = \frac{8}{3}$.

Set $C_2 = f^{-1}([\frac{v_1}{4}, \frac{a}{2} - \frac{v_1}{4}]\backslash\left(\left(\cup_{i=1}^{r_1} V_1^i\right) \cup \left(\cup_{i=1}^{r_2} V_2^i\right)\right))$.

If C_2 is finite or empty then setting $h(\cdot) = 2\left(\varphi_2 \circ d(\ \cdot\ , x) - \varphi_2(d(0,x))\right)$, we obtain $\|h\|_L \leq \frac{16}{3}$, $|h(x) - h(y)| = d(x,y)$, $h(0) = 0$ and with

$$0 < \delta = \begin{cases} v_2/2, & \text{if } C_2 = \emptyset \\ \dfrac{1}{2}\min\left(\{v_2, \operatorname{sep}(C_2)\} \cup \{\operatorname{dist}(z, K\backslash C_2),\ z \in D_2\}\right), & \text{otherwise} \end{cases}$$

where $D_2 = f^{-1}\left([\frac{v_1}{4}, \frac{a}{2} - \frac{v_1}{4}]\backslash\left(\left(\cup_{i=1}^{r_1} \overline{V_1^i}\right) \cup \left(\cup_{i=1}^{r_2} \overline{V_2^i}\right)\right)\right)$.

When $z, t \in K$ are such that $d(z,t) \leq \delta$, then $h(z) = h(t)$, i.e. $h \in lip_0(K)$.

If C_2 is infinite we proceed inductively in a similar way until we get C_n finite, which eventually happens because we have a decreasing sequence of ordinals.

The function h we obtain verifies $h(0) = 0$, $|h(y) - h(x)| = d(x,y)$ and

$$\|h\|_L \leq 2\prod_{j=1}^{n}\left(1 + \frac{1}{2^j - 1}\right) \leq 2\prod_{j=1}^{+\infty}\left(1 + \frac{1}{2^j - 1}\right) := c$$

where c does not depend on x and y. Moreover, setting

$$0<\delta=\begin{cases} v_n/2, & \text{if } C_n=\emptyset \\ \dfrac{1}{2}\left(\min\{v_n, \mathrm{sep}(C_n)\}\cup\{\mathrm{dist}(z,K\backslash C_n),\ z\in D_n\}\right), & \text{otherwise}\end{cases}$$

if $z,t\in K$ are such that $d(z,t)\leq\delta$, then $h(z)=h(t)$, i.e. $h\in lip_0(K)$.

This concludes the proof.

□

3. Metric Approximation Property

Theorem 3.1. *Let (K,d) be a countable compact metric space. Then $\mathcal{F}(K)$ has the metric approximation property.*

Before starting the proof let us recall a construction due to N. Kalton [8].

Let (K,d) be an arbitrary pointed metric space and set

$$K_n=\{x\in K,\ d(0,x)\leq 2^n\} \text{ and } O_n=\{x\in K,\ d(0,x)<2^n\},\ n\in\mathbb{Z}$$
$$F_N=K_{N+1}\backslash O_{-N-1},\ N\in\mathbb{N}.$$

Then, for every $n\in\mathbb{Z}$, we can define a linear operator $T_n:\mathcal{F}(K)\to\mathcal{F}(K)$ by:

$$T_n\delta(x)=\begin{cases} 0 & ,\ x\in K_{n-1} \\ (\log_2 d(0,x)-(n-1))\,\delta(x) & ,\ x\in K_n\backslash K_{n-1} \\ (n+1-\log_2 d(0,x))\,\delta(x) & ,\ x\in K_{n+1}\backslash K_n \\ 0 & ,\ x\notin K_{n+1}\end{cases}.$$

If we set for $N\in\mathbb{N}$, $S_N=\displaystyle\sum_{n=-N}^{N}T_n$, then Lemma 4.2 in [8] gives:

Lemma 3.2. *For every $N\in\mathbb{N}$, we have $\|S_N\|\leq 72$, $S_N(\mathcal{F}(K))\subset\mathcal{F}(F_N)$ and for every $\gamma\in\mathcal{F}(K)$, $\displaystyle\lim_{N\to+\infty}S_N\gamma=\gamma$.*

In order to prove Theorem 3.1 we need the following classical lemma. We will give its proof for sake of completeness.

Lemma 3.3. *If for α countable ordinal there exist $F_1,\cdots,F_n$ clopen subsets of $K^{(\alpha)}$, mutually disjoint, such that $K^{(\alpha)}=F_1\cup\cdots\cup F_n$, then there exist $G_1,\cdots,G_n$ clopen subsets of K, mutually disjoint, such that $K=G_1\cup\cdots\cup G_n$ and $G_i^{(\alpha)}=F_i$.*

Proof: We proceed by induction on $\alpha<\omega_1$ such that $K^{(\alpha)}=F_1\cup\cdots\cup F_n$, for all $1\leq i\neq j\leq n$, F_i is clopen in $K^{(\alpha)}$ and $F_i\cap F_j=\emptyset$.

The result is clear for $\alpha=0$.

Assume that the result is true for $\alpha<\omega_1$ and suppose that $\{F_i\}_{1\leq i\leq n}$ is a clopen partition of $K^{(\alpha+1)}$.

Each F_i is closed in $K^{(\alpha)}$ which is compact, then we can find O_i open subset of $K^{(\alpha)}$ such that $F_i\subset O_i$, $O_i'=F_i$, and $O_i\cap O_j=\emptyset$, for $i\neq j$. Set $O=K^{(\alpha)}\backslash\cup_{i=1}^n O_i$, $U_1=O_1\cup O$ and $U_i=O_i$, for $2\leq i\leq n$. Then $K^{(\alpha)}=\cup_{i=1}^n U_i$, $U_i'=F_i$, and every U_i is clopen in $K^{(\alpha)}$. Indeed we defined $O_i, 2\leq i\leq n$, as open subsets of $K^{(\alpha)}$ so U_i is open in $K^{(\alpha)}$. Moreover points in O are isolated points of $K^{(\alpha)}$ thus O and then U_1 are open in $K^{(\alpha)}$. Finally $K^{(\alpha)}=\cup_{i=1}^n U_i$ then every U_i's are closed.

We can apply the induction hypothesis to find $G_1, \cdots, G_n$ clopen subsets of K, mutually disjoints, such that $K = G_1 \cup \cdots \cup G_n$ and $G_i^{(\alpha)} = U_i$, that is $G_i^{(\alpha+1)} = F_i$, $1 \leq i \leq n$.

Finally we assume α is a limit ordinal and $K^{(\alpha)} = F_1 \cup \cdots \cup F_n$, disjoint union of clopen sets in $K^{(\alpha)}$. There exist $O_1, \cdots, O_n$ open subsets of K such that $F_i \subset O_i$, $O_i^{(\alpha)} = F_i$ and $O_i \cap O_j = \emptyset$ for $i \neq j$.
Set $F = K \setminus \cup_{i=1}^n O_i$, then $\bigcap_{\beta<\alpha} F \cap K^{(\beta)} = F \cap K^{(\alpha)} = \emptyset$. But F is compact, then there exists $\beta < \alpha$ such that $F \cap K^{(\beta)} = \emptyset$, that is to say $K^{(\beta)} \subset \cup_{i=1}^n O_i$. Finally $K^{(\beta)}$ is the disjoint union of $O_i \cap K^{(\beta)}$, $1 \leq i \leq n$, clopen sets in $K^{(\beta)}$, so we can use the induction hypothesis to write $K = G_1 \cup \cdots \cup G_n$, G_i mutually disjoint and clopen in K and $G_i^{(\beta)} = O_i \cap K^{(\beta)} = O_i^{(\beta)}$. Moreover we have $\beta < \alpha$ thus $G_i^{(\alpha)} = \bigcap_{\gamma<\alpha} G_i^{(\gamma)} = \bigcap_{\gamma<\alpha} O_i^{(\gamma)} = O_i^{(\alpha)} = F_i$. $\square$

Proof of Theorem 3.1: We proceed by induction on $\alpha < \omega_1$ such that $K^{(\alpha)}$ is finite.

- If K is finite then $\mathcal{F}(K)$ is finite dimensional, so has trivially the MAP and the property is true for $\alpha = 0$.
- Let α be a countable ordinal and assume that for every $\beta < \alpha$, if (K, d) is a compact metric space so that $K^{(\beta)}$ is finite, then $\mathcal{F}(K)$ has the MAP.

  Now let (K, d) be a compact metric space such that $K^{(\alpha)}$ is finite.

  First $\mathcal{F}(K)$ is linearly isometric to $lip_0(K)^*$ and a theorem of A. Grothendieck [6] (see also Theorem 1.e.15 in [10]) asserts that a separable Banach space which is isometric to a dual space and which has the AP has the MAP, so it is enough to prove that $\mathcal{F}(K)$ has the BAP.

  Secondly, if K is such that $K^{(\alpha)} = \{a_1, \cdots, a_n\}$, singletons $\{a_i\}$'s are clopen in $K^{(\alpha)}$ and Lemma 3.3 gives $G_1, \cdots, G_n$ mutually disjoint clopen subsets of K such that $\forall i \leq n$, $G_i^{(\alpha)} = \{a_i\}$ and $K = G_1 \cup \cdots \cup G_n$. Moreover $\mathcal{F}(K)$ is isomorphic to $(\oplus_{i=1}^n \mathcal{F}(K_i))_{\ell_1}$, where $K_i = G_i \cup \{0\}$, $1 \leq i \leq n$.

  Indeed if $a = \min\limits_{i \neq j} \operatorname{dist}(G_i, G_j)$, where

  $$\operatorname{dist}(G_i, G_j) = \inf \{d(x,y) \ , \ x \in G_i \ , \ y \in G_j\},$$

  by compactness we have $a > 0$. Then the operator

  $$\Phi : \begin{array}{ccc} Lip_0(K) & \to & (\oplus_{i=1}^n Lip_0(K_i))_\infty \\ f & \mapsto & \left(f_{|K_i}\right)_{i=1}^n \end{array}$$

  is onto, linear, weak*-continuous and for $f \in Lip_0(K)$, we have

  $$\frac{a}{2 \operatorname{diam}(K)} \|f\|_L \leq \|\Phi(f)\|_\infty \leq \|f\|_L.$$

  Hence $\mathcal{F}(K)$ is isomorphic to $(\oplus_{i=1}^n \mathcal{F}(K_i))_{\ell_1}$.

  The BAP is stable with respect to finite ℓ_1−sums and isomorphisms, then it is enough to prove that for any $i \in \{1, \cdots, n\}$, $\mathcal{F}(K_i)$ has the BAP. In other words we need to prove that when $K^{(\alpha)}$ is a singleton then $\mathcal{F}(K)$ has the BAP.

Suppose as we may that $K^{(\alpha)} = \{0\}$. Using the construction due to Kalton [8] we have a sequence of linear operators $S_N : \mathcal{F}(K) \to \mathcal{F}(F_N)$, $\|S_N\| \leq 72$ and for every $\gamma \in \mathcal{F}(K)$, $\lim\limits_{N\to+\infty} S_N\gamma = \gamma$.

Moreover, for every $N \in \mathbb{N}$, there exists $\beta < \alpha$ such that $F_N^{(\beta)}$ is finite and then $\mathcal{F}(F_N)$ has the MAP:

since $\mathcal{F}(F_N)$ is separable, for every $N \in \mathbb{N}$, there exists a sequence of finite-rank linear operators $R_p^N : \mathcal{F}(F_N) \to \mathcal{F}(F_N)$ so that for every $\gamma \in \mathcal{F}(F_N)$, $\lim\limits_{p\to+\infty} R_p^N\gamma = \gamma$ and $\|R_p^N\| \leq 1$ for every $p \in \mathbb{N}$ ([12], see also Theorem 1.e.13 in [10]).

Setting $Q_{N,p} = R_p^N \circ S_N$ we deduce that the range of $Q_{N,p}$ is finite dimensional as the range of R_p^N, $\|Q_{N,p}\| \leq \|R_p^N\|\|S_N\| \leq 72$ and for every $\gamma \in \mathcal{F}(K)$,

$$\lim_{N\to+\infty}\lim_{p\to+\infty} R_p^N S_N\gamma = \lim_{N\to+\infty} S_N\gamma = \gamma.$$

Thus $\mathcal{F}(K)$ has the 72-BAP and this concludes the proof.

□

4. Application to quotient spaces

For a pointed metric space (M,d) and A a closed subset of M containing 0 we can define the quotient M/A as the space $(M\backslash A)\cup\{0\}$ with the metric given by:

$$d_{M/A}(x,y) = \begin{cases} \operatorname{dist}(x,A) & ,\ y=0 \\ \min\{d(x,y),\ \operatorname{dist}(x,A)+\operatorname{dist}(y,A)\} & ,\ x,y\neq 0 \end{cases}.$$

Corollary 4.1. Let (K,d) be a compact metric space which is not perfect (i.e. $K' \neq K$). Then for every countable ordinal $\alpha \geq 1$, the space $\mathcal{F}(K/K^{(\alpha)})$ has the MAP.

Proof: Remark that for every compact metric space (K,d) and every countable ordinal $\alpha \geq 1$, the quotient space $K/K^{(\alpha)}$ is compact and countable because $\left(K/K^{(\alpha)}\right)^{(\alpha)}$ is empty or $\{0\}$. Then this result is a consequence of Theorem 3.1. □

Remark 4.2. (1) If K is perfect, then $\mathcal{F}(K/K^{(\alpha)}) = \{0\}$.

(2) Otherwise $\mathcal{F}(K)/\mathcal{F}(K^{(\alpha)})$ is linearly isometric to $\mathcal{F}(K/K^{(\alpha)})$ (We write $\mathcal{F}(K)/\mathcal{F}(K^{(\alpha)}) \equiv \mathcal{F}(K/K^{(\alpha)})$).

Indeed we can assume that $0 \in K^{(\alpha)}$. Then

$$\begin{aligned} &\{f \in Lip_0(K)\ ;\ \forall x,y \in K^{(\alpha)}, f(x)=f(y)\} \\ &= \{f \in Lip_0(K)\ ;\ \forall x \in K^{(\alpha)}, f(x)=0\}. \end{aligned}$$

And since $\mathcal{F}(K^{(\alpha)}) = \overline{\text{vect}}\ \{\delta_x,\ x \in K^{(\alpha)}\}$, we have

$$\{f \in Lip_0(K)\ ;\ \forall x \in K^{(\alpha)}, f(x)=0\} = \mathcal{F}(K^{(\alpha)})^{\perp},$$

which is isometric to $\left(\mathcal{F}(K)/\mathcal{F}(K^{(\alpha)})\right)^*$. To sum up

$$\{f \in Lip_0(K)\ ;\ \forall x,y \in K^{(\alpha)}, f(x)=f(y)\} \equiv \Big(\mathcal{F}(K)/\mathcal{F}(K^{(\alpha)})\Big)^*.$$

From Propositions 1.4.3 and 1.4.4 in [14], there exists an isometry Φ from $\{f \in Lip_0(K)\ ;\ \forall x,y \in K^{(\alpha)}, f(x)=f(y)\}$ onto $Lip_0\left(K/K^{(\alpha)}\right)$.

Moreover $Lip_0(K/K^{(\alpha)})$ is linearly isometric to $\mathcal{F}(K/K^{(\alpha)})^*$, so the space $\left(\mathcal{F}(K)/\mathcal{F}(K^{(\alpha)})\right)^*$ is isomorphically isometric to $\mathcal{F}(K/K^{(\alpha)})^*$. One can easily check that Φ is weak*-continuous, and finally $\mathcal{F}(K)/\mathcal{F}(K^{(\alpha)})$ is linearly isometric to $\mathcal{F}(K/K^{(\alpha)})$.

To finish this paper we will use Corollary 4.1 and the previous remark to prove the following: in order to obtain that every countable compact metric space has the BAP it is not possible to use the three-space property due to G. Godefroy and P.D. Saphar [5], asserting:

If M is a closed subspace of a Banach space X so that $M^\perp$ is complemented in X^* and X/M has the BAP, then X has the BAP if and only if M has the BAP.

Indeed we can construct a compact metric space K so that $K^{(2)} = \{0\}$, in particular $\mathcal{F}(K)$, $\mathcal{F}(K')$ and $\mathcal{F}(K)/\mathcal{F}(K')$ have the MAP, but $\mathcal{F}(K')^\perp$ is not complemented in $Lip_0(K)$.

To construct this space we need a proposition similar to Proposition 7 in [4]:

Proposition 4.3. For any $\lambda > 0$, there exist a finite metric space H_λ and a subset G_λ of H_λ such that if $P : Lip_0(H_\lambda) \to \mathcal{F}(G_\lambda)^\perp$ is a bounded linear projection, then $\|P\| \geq \lambda$.

Proof: Assume that for some $\lambda_0 > 0$ and for all pairs (G, H) of finite metric spaces with $G \subset H$ we can construct $P : Lip_0(H) \to \mathcal{F}(G)^\perp$ linear projection with norm bounded by λ_0.

Let K be the compact metric space such that $\mathcal{F}(K)$ fails AP appearing in Corollary 5 of [4]. There exists $(G_n)_{n\in\mathbb{N}}$ an increasing sequence of finite subsets of K such that $\overline{\bigcup_{n\in\mathbb{N}} G_n} = K$.

Then for every $n \in \mathbb{N}$ and every $k \geq n$, there exists $P_n^k : Lip_0(G_k) \to \mathcal{F}(G_n)^\perp$ a linear projection of norm less than λ_0, where $\mathcal{F}(G_n)^\perp \subset Lip_0(G_k)$.

Fix $n \in \mathbb{N}$. For $k \in \mathbb{N}$, let $E_k : Lip_0(G_k) \to Lip_0(K)$ be the non linear extension operator which preserves the Lipschitz constant given by the inf-convolution formula:

$$\forall f \in Lip_0(K),\ \forall x \in K,\ E_k f(x) = \inf_{y\in G_k} \{f(y) + \|f\|_L d(x, y)\}.$$

For $f \in Lip_0(K)$, we set

$$\widetilde{P_n^k}(f) = \begin{cases} E_k P_n^k\left(f_{|G_k}\right) & ,k \geq n \\ 0 & ,k < n \end{cases}.$$

Then $\|\widetilde{P_n^k}(f)\|_L \leq \lambda_0 \|f\|_L$, for every $f \in Lip_0(K)$.

If $\mathcal{U}$ is a non trivial ultrafilter on $\mathbb{N}$, for every $f \in Lip_0(K)$ we can define $P_n f$ as the pointwise limit of $\widetilde{P_n^k}(f)$ with respect to $k \in \mathcal{U}$. Then P_n is a linear projection onto $\mathcal{F}(G_n)^\perp \subset Lip_0(K)$ because P_n^k is a projection onto $\mathcal{F}(G_n)^\perp \subset Lip_0(G_k)$. Moreover $\|P_n f\|_L \leq \lambda_0 \|f\|_L$ and $P_n f$ pointwise converges to 0 for any $f \in Lip_0(K)$.

Set $Q_n = Id_{Lip_0(K)} - P_n : Lip_0(K) \to Lip_0(K)$. Then Q_n is a continuous linear projection of finite rank and Ker $Q_n = \mathcal{F}(G_n)^\perp$ is weak*-closed. Therefore Q_n is weak*-continuous. Moreover $\|Q_n\| \leq 1 + \lambda_0$ and for every $f \in Lip_0(K)$, $Q_n f$ converges pointwise to f.

Using Theorem 2 in [1] we deduce that $\mathcal{F}(K)$ has the $(1+\lambda_0)$-BAP, contradicting our assumption on K.

□

Thanks to that proposition we will construct a compact metric space K such that $K^{(2)} = \{0\}$ and $\mathcal{F}(K')^{\perp}$ is not complemented in $Lip_0(K)$:

For every $n \in \mathbb{N}$ there exist $A_n \subset B_n$ finite such that for every continuous linear projection $P_n : Lip_0(B_n) \to \mathcal{F}(A_n)^{\perp}$, we have $\|P_n\| \geq n$.

Set $\alpha_n = \min\{d(x,y)\ ,\ x \neq y \in B_n\} > 0$. If we see B_n as a subspace of $\ell_\infty^{m_n}$, with m_n the cardinality of B_n, we can find for every $a \in A_n$, L_n^a a sequence converging to a such that $L_n^a \subset B\left(a, \frac{\alpha_n}{2}\right)$.

Define $K_n = \left(\bigcup_{a \in A_n} L_n^a\right) \cup B_n$, we obtain $A_n \subset B_n \subset K_n$ and $K_n' = A_n$. We can assume that the diameter of K_n is less than 8^{-n}.

Finally we define $K := \left(\bigcup_{n \in \mathbb{N}} \{n\} \times K_n\right) \cup \{0\}$ equipped with the distance:

$$\begin{aligned} d(0,(n,x)) &= 2^{-n} \\ d((n,x),(m,y)) &= \begin{cases} d_{K_n}(x,y) & , n = m \\ |2^{-n} - 2^{-m}| & , n \neq m \end{cases} . \end{aligned}$$

Then $K^{(2)} = \{0\}$.

Now assume that there exists $P : Lip_0(K) \to \mathcal{F}(K')^{\perp}$ a continuous linear projection. Let $E_n : Lip_0(B_n) \to Lip_0(K_n)$, $F_n : Lip_0(B_n) \to Lip_0(K)$ and $R_n : Lip_0(K) \to Lip_0(B_n)$ be defined as follows:

$$\begin{aligned} \forall f \in Lip_0(B_n), \qquad (E_n f)(x) &= \begin{cases} f(x) & , x \in B_n \\ f(a) & , x \in L_n^a \end{cases} \\ \forall f \in Lip_0(B_n), \quad (F_n f)(m,x) &= \begin{cases} (E_n f)(x) & , n = m \\ 0 & , n \neq m \end{cases} . \\ \forall f \in Lip_0(K), \qquad R_n f &= f_{|\{n\} \times B_n} \end{aligned}$$

We set $P_n := R_n \circ P \circ F_n : Lip_0(B_n) \to \mathcal{F}(A_n)^{\perp}$ and we have that P_n is a continuous linear projection. From Proposition 4.3 we deduce that $\|P_n\| \geq n$. Moreover our choice of α_n implies that $\|F_n\| \leq 3$, then finally $\|P\| \geq n/3$ holds for every $n \in \mathbb{N}$. Therefore P is unbounded and $\mathcal{F}(K')^{\perp}$ is not complemented in $Lip_0(K)$.

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