# FREE SPACES OVER SOME PROPER METRIC SPACES

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Abstract. We prove that the Lipschitz-free space over a countable proper metric space is isometric to a dual space and has the metric approximation property. We also show that the Lipschitz-free space over a proper ultrametric space is isometric to the dual of a space which is isomorphic to $c_0(\mathbb{N})$.

## 1. Introduction

For a pointed metric space $(M,d)$, that is a metric space with an origin 0, we denote $Lip_0(M)$ the space of Lipschitz real-valued functions on $M$ which vanish at 0. Endowed with the norm defined by the Lipschitz constant, this space is a Banach space. Moreover, its unit ball is compact for the pointwise topology, hence it is a dual space.

Let $x \in M$ and define $\delta_x \in Lip_0(M)^*$ as follows: for $f \in Lip_0(M)$, $\delta_x(f) = f(x)$. The Lipschitz-free space over $M$, denoted $\mathcal{F}(M)$, is the closed subspace of $Lip_0(M)^*$ spanned by the $\delta_x$'s: $\mathcal{F}(M) := \overline{\operatorname{span}}\{\delta_x,\ x \in M\}$. Its dual space is isometrically isomorphic to $Lip_0(M)$.

Lipschitz-free spaces are considered in [20], where they are called Arens-Eells spaces. The notation we use is due to Godefroy and Kalton [6] where they point out that despite the simplicity of the definition of $\mathcal{F}(M)$, it is not easy to study its linear structure. Although their article was published in 2003, still very little is known about Lipschitz-free spaces. One can check that the Lipschitz-free space over $\mathbb{R}$ is $L_1(\mathbb{R})$, but Naor and Schechtman proved in [17] that $\mathcal{F}(\mathbb{R}^2)$ is not isomorphic to a subspace of any $L_1$. Moreover, Godard [4] proved that $\mathcal{F}(M)$ is isometrically isomorphic to a subspace of an $L_1$-space if and only if $M$ isometrically embeds into an $\mathbb{R}$-tree. We will focus on the notion of approximation property.

A Banach space $X$ has the approximation property (AP in short) if for every positive $\varepsilon$, every $K \subset X$ compact, there exists an operator $T$ on $X$, of finite rank, such that for every $x \in K$, the norm $\|Tx - x\|$ is less than $\varepsilon$.

Let $\lambda \in [1, +\infty)$. The space $X$ has the $\lambda$-bounded approximation property ($\lambda$-BAP) if for every positive $\varepsilon$, every $K \subset X$ compact, there exists an operator $T$ on $X$, of finite rank, such that $\|T\| \leq \lambda$ and for every $x \in K$, the norm $\|Tx - x\|$ is less than $\varepsilon$.

Finally, $X$ has the metric approximation property (MAP) when it has the 1-BAP.

Godefroy and Kalton [6] proved that a Banach space has the $\lambda$-BAP if and only if its Lipschitz-free space has the $\lambda$-BAP. Lancien and Pernecká [13] proved that the Lipschitz-free space over a doubling metric space has the BAP and that $\mathcal{F}(\ell_1)$ has a finite-dimensional Schauder decomposition. Hájek and Pernecká improved this last result, in [9] they obtained that $\mathcal{F}(\ell_1)$ has a Schauder basis. However, there

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are not only positive results, Godefroy and Ozawa [7] constructed a compact metric space $(K,d)$ such that $\mathcal{F}(K)$ fails the AP. But the author proved in [1] that in the case of countable compact metric spaces, the Lipschitz-free space always has the MAP. In this article, we will prove that the Lipschitz-free space over a countable proper metric space and over a proper ultrametric space is a dual space and has the MAP. More precisely, we show that in the case of a proper ultrametric space, the Lipschitz-free space has an isometric predual which is isomorphic to $c_0(\mathbb{N})$.

## 2. Countable proper metric spaces

A metric space is said to be proper if every closed ball is compact.

For a metric space $(M,d)$, we will denote by $B(x,r)$ the open ball of center $x \in M$ and radius $r>0$, and by $\overline{B}(x,r)$ the closed ball.

The space $lip_0(M)$ is the subspace of $Lip_0(M)$ of functions $f$ satisfying:

$$\forall \varepsilon>0, \exists \delta>0: \ d(x,y)<\delta \Rightarrow |f(x)-f(y)| \leq \varepsilon d(x,y)$$

The first result of this section is the following:

*Theorem* 2.1. Let $M$ be a countable proper metric space and

$$S=\left\{ f\in lip_0(M);\ \lim_{r\to+\infty} \sup_{\substack{x \text{ or } y\notin \overline{B}(0,r)\\ x\neq y}} \frac{f(x)-f(y)}{d(x,y)}=0 \right\}.$$

Then, $\mathcal{F}(M)$ is isometrically isomorphic to $S^*$.

Before the proof, we need some definitions:

**Definition 2.2.**

(1) Let $X$ be a Banach space. A subspace $F$ of $X^*$ is called separating if $x^*(x)=0$ for all $x^*\in F$ implies $x=0$.
(2) For $(M,d)$ a pointed metric space, a subspace $F$ of $Lip_0(M)$ separates points uniformly if there exists a constant $c\geq 1$ such that for every $x,y\in M$, some $f\in F$ satisfies $\|f\|_L\leq c$ and $|f(x)-f(y)|=d(x,y)$.

**Definition 2.3.** Let $X$ be a Banach space. We denote $NA(X)$ the subset of $X^*$ consisting of all linear forms which attain their norm.

A result of Petunīn and Plīčko [19] asserts that for a separable Banach space $X$, if a closed subspace $F$ of $X^*$ is separating and is a subset of $NA(X)$, then $X$ is isometrically isomorphic to $F^*$. To use this result we proceed with a few lemmas about the space $S$.

*Lemma* 2.4. Let $(M,d)$ be proper pointed metric space. The space $S$ is a subspace of $NA(\mathcal{F}(M))$.

*Proof.* Let $f\in S$. We may assume that $f\neq 0$ and take $0<\varepsilon<\frac{\|f\|_L}{2}$. Since

$$\lim_{r\to+\infty} \sup_{\substack{x \text{ or } y\notin \overline{B}(0,r)\\ x\neq y}} \frac{f(x)-f(y)}{d(x,y)}=0$$

there exists $r > 0$ such that

$$\sup_{\substack{x \text{ or } y \notin \overline{B}(0,r) \\ x \neq y}} \frac{|f(x) - f(y)|}{d(x,y)} < \varepsilon.$$

Thus, $\|f\|_L = \sup\limits_{\substack{x,y \in \overline{B}(0,r) \\ y \neq x}} \dfrac{|f(x) - f(y)|}{d(x,y)}$.

Because $f \in lip_0(M)$, the set

$$\overline{B}_\varepsilon^2 := \big\{(x,y) \in \overline{B}(0,r)^2,\ x \neq y,\ |f(x) - f(y)| \geq \varepsilon\, d(x,y)\big\}$$

is compact and we have

$$\begin{aligned} \|f\|_L = \sup_{\substack{x,y \in \overline{B}(0,r) \\ y \neq x}} \frac{|f(x) - f(y)|}{d(x,y)} &= \sup_{(x,y) \in \overline{B}_\varepsilon^2} \frac{|f(x) - f(y)|}{d(x,y)} \\ &= \max_{(x,y) \in \overline{B}_\varepsilon^2} \frac{|f(x) - f(y)|}{d(x,y)}. \end{aligned}$$

Thus, there exist $x \neq y$ such that $\|f\|_L = \frac{|f(x)-f(y)|}{d(x,y)}$. With $\gamma = \frac{1}{d(x,y)}(\delta_x - \delta_y)$, $\gamma \in \mathcal{F}(M)$, we obtain $\|f\|_L = |f(\gamma)|$, with $\|\gamma\|_{\mathcal{F}(M)} = 1$ because $\delta$ is an isometry. Then, $f$ is norm attaining and $S \subset NA(\mathcal{F}(M))$. □

*Lemma* 2.5. Let $(M,d)$ be a proper pointed metric space. If $S$ separates points uniformly, then it is separating.

*Proof.* Using Hahn-Banach theorem it is enough to prove that when $S$ separates points uniformly, it is weak$^*$-dense in $Lip_0(M)$.

We will first prove that the condition

$$\lim_{r \to +\infty} \sup_{\substack{x \text{ or } y \notin \overline{B}(0,r) \\ x \neq y}} \frac{f(x) - f(y)}{d(x,y)} = 0$$

is stable under supremum and infimum between two functions.

Let $f, g \in S$ and $x \neq y$ in $M$ such that $x$ or $y$ doesn't belong to $\overline{B}(0,r)$. We assume that $f(x) \leq g(x)$, the other case is similar. We need to distinguish two cases:

- if $f(y) \leq g(y)$, then

$$\frac{\inf(f,g)(x) - \inf(f,g)(y)}{d(x,y)} = \frac{f(x) - f(y)}{d(x,y)}$$

and

$$\frac{\sup(f,g)(x) - \sup(f,g)(y)}{d(x,y)} = \frac{g(x) - g(y)}{d(x,y)}$$

- if $f(y) \geq g(y)$, then

$$\frac{f(x) - f(y)}{d(x,y)} \leq \frac{\inf(f,g)(x) - \inf(f,g)(y)}{d(x,y)} = \frac{f(x) - g(y)}{d(x,y)} \leq \frac{g(x) - g(y)}{d(x,y)}$$

and

$$\frac{f(x)-f(y)}{d(x,y)} \leq \frac{\sup(f,g)(x)-\sup(f,g)(y)}{d(x,y)} = \frac{g(x)-f(y)}{d(x,y)} \leq \frac{g(x)-g(y)}{d(x,y)}$$

So we obtain:

$$\lim_{r\to+\infty} \sup_{\substack{x \text{ or } y\notin \overline{B}(0,r)\\ x\neq y}} \frac{\inf(f,g)(x)-\inf(f,g)(y)}{d(x,y)} = 0$$

and

$$\lim_{r\to+\infty} \sup_{\substack{x \text{ or } y\notin \overline{B}(0,r)\\ x\neq y}} \frac{\sup(f,g)(x)-\sup(f,g)(y)}{d(x,y)} = 0$$

finally $\inf(f,g), \sup(f,g) \in S$.

Assume that $M$ is a proper pointed metric space such that $S$ separates points uniformly. Mimicking the proof of Lemma 3.2.3 in [20] we obtain the following: there exists $b \geq 1$ such that for all $f \in Lip_0(M)$, for all $A$ finite subset of $M$ containing 0, we can find $g \in S$ so that $\|g\|_L \leq b\|f\|_L$ and $g_{|A} = f_{|A}$. Finally, one can deduce that the weak*-closure of $S$ is $Lip_0(M)$, so $S$ is separating. □

Along the proof of Theorem 2.1 we will need a characterization of compact metric spaces which are countable. First define the Cantor-Bendixon derivation. For a metric space $(M,d)$ we denote:

- $M'$ the set of accumulation points of $M$.
- $M^{(\alpha)} = (M^{(\alpha-1)})'$, for a successor ordinal $\alpha$.
- $M^{(\alpha)} = \bigcap\limits_{\beta<\alpha} M^{(\beta)}$, for a limit ordinal $\alpha$.

A compact metric space $(K,d)$ is countable if and only if there is a countable ordinal $\alpha$ such that $K^{(\alpha)}$ is finite.

*Proof of Theorem 2.1:* Note first that the subspace $S$ of $\mathcal{F}(M)^*$ defined previously is closed in $\mathcal{F}(M)^*$, so it follows from Lemmas 2.4, 2.5 and from Petunīn and Plīčko's result [19] that we only have to prove that $S$ separates points uniformly.

Ideas are the same as in the proof of Theorem 2.1 in [1] but for sake of completeness we will give all details. Let $M$ be a proper countable metric space, $x, y \in M$ and $a = d(x,y)$. The ball $\overline{B}\left(x, \frac{3a}{2}\right)$ is compact and countable so there exist a countable ordinal $\alpha_0$, $k_1 \in \mathbb{N}$ and $y_1^1, \cdots, y_{k_1}^1 \in M$ such that

$$\overline{B}\left(x, \frac{3a}{2}\right)^{(\alpha_0)} = \{y_1^1, \cdots, y_{k_1}^1\}$$

We can find $r_1, s_1, t_1$ and

$$u_1^1 < \cdots < u_{r_1}^1 \leq \frac{a}{2} < v_1^1 < \cdots < v_{s_1}^1 < a \leq w_1^1 < \cdots < w_{t_1}^1 \leq \frac{3a}{2}$$

such that $\{d(x,y_i^1), 1 \leq i \leq k_1\} = \{u_1^1, \cdots, u_{r_1}^1, v_1^1, \cdots, v_{s_1}^1, w_1^1, \cdots, w_{t_1}^1\}$. Now set

$$u_1 = \min\left( \begin{array}{c} \left\{\frac{a}{2},\ u_1^1,\ \frac{a}{2}-u_{r_1}^1,\ w_1^1 - a,\ \frac{3a}{2} - w_{t_1}^1\right\} \setminus \{0\} \\ \bigcup \left\{u_1^i - u_1^{i-1},\ 2 \leq i \leq r_1\right\} \cup \left\{w_1^i - w_1^{i-1},\ 2 \leq i \leq t_1\right\} \end{array} \right)$$

and define $\varphi_1 : [0,+\infty) \to [0,+\infty)$ by

$$\varphi_1(t)=\left\{\begin{array}{ll} 0 & ,\ t\in\left[0,\frac{u_1}{4}\right):=U_1^0\\ u_1^i & ,\ t\in\left(u_1^i-\frac{u_1}{4},u_1^i+\frac{u_1}{4}\right):=U_1^i,\ 1\le i\le r_1\\ \frac{a}{2} & ,\ t\in\left(\frac{a}{2}-\frac{u_1}{4},a+\frac{u_1}{4}\right):=W_1^0\ (\text{possibly }\ U_1^{r_1},W_1^1\subset W_1^0)\\ \frac{3a}{2}-w_1^i, & t\in\left(w_1^i-\frac{u_1}{4},w_1^i+\frac{u_1}{4}\right):=W_1^i,\ 1\le i\le t_1\\ 0 & ,\ t\in\left(3\frac{a}{2}-\frac{u_1}{4},+\infty\right):=W_1^{t_1+1}\ (\text{possibly }\ W_1^{t_1}\subset W_1^{t_1+1}) \end{array}\right.$$

and $\varphi_1$ is affine on each interval of $[0,+\infty)\setminus\left(\left(\cup_{i=0}^{r_1}U_1^i\right)\cup\left(\cup_{i=0}^{t_1+1}W_1^i\right)\right)$. One can check that $\|\varphi_1\|_L\le 2$.

With $f(\cdot)=d(\cdot,x)$, we set

$$C_1=f^{-1}\left([0,+\infty)\setminus\left(\left(\cup_{i=0}^{r_1}U_1^i\right)\cup\left(\cup_{i=0}^{t_1+1}W_1^i\right)\right)\right).$$

First, if $C_1$ is finite or empty, define $h(\cdot)=2\left(\varphi_1\circ d(\cdot,x)-\varphi_1\circ d(0,x)\right)$. Then we have $|h(x)-h(y)|=d(x,y)$, $h(0)=0$ and $\|h\|_L\le 4$. We need to prove that $h\in lip_0(M)$. Set

$$\delta=\left\{\begin{array}{ll} u_1/2, & \text{if } C_1=\emptyset\\ \dfrac{1}{2}\inf\left(\{u_1,\mathrm{sep}(C_1)\}\cup\{\mathrm{dist}(z,M\backslash C_1),\ z\in D_1\}\right), & \text{otherwise}\end{array}\right.$$

where

$$\mathrm{sep}(C_1)=\inf\{d(z,t),\ z\neq t,\ z,t\in C_1\}$$

and

$$D_1=f^{-1}\left([0,+\infty)\setminus\left(\left(\cup_{i=0}^{r_1}\overline{U_1^i}\right)\cup\left(\cup_{i=0}^{t_1+1}\overline{W_1^i}\right)\right)\right).$$

Since $C_1$ is finite we have $\mathrm{sep}(C_1)>0$. Moreover, $\mathrm{dist}(z,M\backslash C_1)>0$ for $z\in D_1$ and $D_1$ is finite. Thus we deduce that $\delta>0$.

If follows that every $z\neq t\in M$ such that $d(z,t)\le\delta$ are not in $D_1$ and there exists $0\le i\le r_1$ such that $z,t\in f^{-1}\left(\overline{U_1^i}\right)$ or $0\le i\le t_1+1$ such that $z,t\in f^{-1}\left(\overline{W_1^i}\right)$, so the equality $h(z)=h(t)$ holds, i.e. $h\in lip_0(M)$.

Finally, let us prove that $\displaystyle\lim_{r\to+\infty}\sup_{\substack{x\text{ or }y\notin\overline{B}(0,r)\\ x\neq y}}\frac{h(x)-h(y)}{d(x,y)}=0$, that is $h\in S$.

Let $r>0$ be such that $\overline{B}(x,\frac{3a}{2})\subset\overline{B}(0,r)$ and $z\notin\overline{B}(0,r)$.

First if $t\notin\overline{B}\left(x,\frac{3a}{2}\right)$ then $h(z)=h(t)$ and $\dfrac{|h(z)-h(t)|}{d(z,t)}=0$.

Secondly if $t\in\overline{B}(x,\frac{3a}{2})$, then

$$\frac{|h(z)-h(t)|}{d(z,t)}=\frac{|h(t)|}{d(z,t)}\le\frac{d(x,y)}{d(z,t)}\underset{r\to+\infty}{\longrightarrow}0$$

so $h\in S$.

Assume now that $C_1$ is infinite. It is a subset of $\overline{B}\left(x,\frac{3a}{2}\right)$ thus for every ordinal $\alpha$ we have $C_1^{(\alpha)}\subset\overline{B}\left(x,\frac{3a}{2}\right)^{(\alpha)}$. Moreover, $C_1\cap\overline{B}(x,\frac{3a}{2})^{(\alpha_0)}=\emptyset$ so we have $C_1^{(\alpha_0)}=\emptyset$. Since $C_1$ is compact and countable we can find $\alpha_1<\alpha_0$ such that $C_1^{(\alpha_1)}$ is finite and non empty.

There exist $k_2\in\mathbb{N}$ and $y_2^1,\cdots,y_2^{k_2}\in C_1$ such that $C_1^{(\alpha_1)}=\{y_2^1,\cdots,y_2^{k_2}\}$. Then we can find $r_2,t_2\in\mathbb{N}$ and

$$u_2^1<\cdots<u_2^{r_2}<\frac{a}{2}-\frac{u_1}{4},\ \frac{3a}{2}+\frac{u_1}{4}<w_2^1<\cdots<w_2^{t_2}$$

such that

$$\{d(x,y_2^i)\ ;\ 1\leq i\leq k_2\}=\{u_2^1,\cdots,u_2^{r_2},w_2^1,\cdots,w_2^{t_2}\}.$$

Set

$$u_2=\min\left(\begin{array}{l}\{u_1,(\frac{a}{2}-\frac{u_1}{4})-u_2^{r_2},w_2^1-(\frac{3a}{2}+\frac{u_1}{4})\}\\ \bigcup\{u_2^i-u_2^{i-1},2\leq i\leq r_2\}\cup\{w_2^i-w_2^{i-1},2\leq i\leq t_2\}\end{array}\right)$$

and define $\varphi_2:[0,+\infty)\to[0,+\infty)$ by

$$\varphi_2(t)=\begin{cases}\varphi_1(t) & ,\ t\in\left(\cup_{i=0}^{r_1}U_1^i\right)\cup\left(\cup_{i=0}^{t_1+1}W_1^i\right)\\ \varphi_1(u_2^i) & ,\ t\in\left(u_2^i-\frac{u_2}{2^3},u_2^i+\frac{u_2}{2^3}\right):=U_2^i,\ 1\leq i\leq r_2\\ \varphi_1(w_2^i) & ,\ t\in\left(w_2^i-\frac{u_2}{2^3},w_2^i+\frac{u_2}{2^3}\right):=W_2^i,\ 1\leq i\leq t_2\end{cases}$$

and $\varphi_2$ is continuous on $[0,+\infty)$ and affine on each interval of

$$[0,+\infty)\backslash\left(\left(\cup_{i=0}^{r_1}U_1^i\right)\cup\left(\cup_{i=0}^{t_1+1}W_1^i\right)\cup\left(\cup_{i=1}^{r_2}U_1^i\right)\cup\left(\cup_{i=1}^{t_2}W_1^i\right)\right).$$

It is easy to check that $\|\varphi_2\|_L\leq\frac{8}{3}$.

Now we set $C_2=C_1\backslash f^{-1}\left(\left(\cup_{i=1}^{r_2}U_1^i\right)\cup\left(\cup_{i=1}^{t_2}W_1^i\right)\right)$. First if $C_2$ is finite or empty the function $h(\cdot)=2\left(\varphi_2\circ d(\cdot,x)-\varphi_2\circ d(0,x)\right)$ verifies $h(0)=0$, $|h(x)-h(y)|=d(x,y)$ and $\|h\|_L\leq\frac{16}{3}$. Moreover, if we set

$$\delta=\begin{cases}u_2/2, & \text{if } C_2=\emptyset\\ \dfrac{1}{2}\min\left(\{u_2,\mathrm{sep}(C_2)\}\cup\{\mathrm{dist}(z,M\backslash C_2),\ z\in D_2\}\right), & \text{otherwise}\end{cases}$$

where $D_2=C_1\backslash f^{-1}\left(\left(\cup_{i=1}^{r_2}\overline{U_1^i}\right)\cup\left(\cup_{i=1}^{t_2}\overline{W_1^i}\right)\right)$, we obtain that $\delta>0$ and when $z,t\in M$ are such that $d(z,t)\leq\delta$, then $h(z)=h(t)$. So finally $h$ is in $lip_0(M)$. The proof of the fact that $h$ belongs to $S$ is the same as previously.

If $C_2$ is infinite we proceed inductively until we get $C_n$ finite, which eventually happens because we construct a decreasing sequence of ordinals.

The function $h$ we finally obtain verifies $h(0)=0$, $|h(y)-h(x)|=d(x,y)$ and

$$\|h\|_L\leq 2\prod_{j=1}^{n}\left(1+\frac{1}{2^j-1}\right)\leq 2\prod_{j=1}^{+\infty}\left(1+\frac{1}{2^j-1}\right):=c$$

where $c$ does not depend on $x$ and $y$. Moreover, setting

$$\delta=\begin{cases}u_n/2, & \text{if } C_n=\emptyset\\ \dfrac{1}{2}\left(\min\{u_n,\mathrm{sep}(C_n)\}\cup\{\mathrm{dist}(z,M\backslash C_n),\ z\in D_n\}\right), & \text{otherwise}\end{cases}$$

we get $\delta>0$ and if $z,t\in M$ are such that $d(z,t)\leq\delta$, then $h(z)=h(t)$, i.e. $h\in lip_0(M)$. Finally, $h$ still verifies $\lim\limits_{r\to+\infty}\sup\limits_{\substack{x\text{ or }y\notin\overline{B}(0,r)\\ x\neq y}}\dfrac{h(x)-h(y)}{d(x,y)}=0$ so we can conclude that $S$ separates points uniformly and therefore $\mathcal{F}(M)$ is isometrically isomorphic to $S^*$. □

We can now prove the second result of this section:

*Theorem* 2.6. The Lipschitz-free space over a countable proper metric space has the metric approximation property.

*Proof:* A theorem of A. Grothendieck [8] asserts that if a separable Banach space is isometrically isomorphic to a dual space and has the AP, then it has the MAP. Thus it follows from Theorem 2.1 that it is enough to prove that for $M$ a countable proper metric space, $\mathcal{F}(M)$ has the BAP.

We need the following result we can deduce from Lemma 4.2 in [11]:
For any pointed metric space $M$, define for $N \in \mathbb{N}$,

$$A_N = \left(\overline{B}(0, 2^{N+1}) \backslash B(0, 2^{-N-1})\right) \cup \{0\}.$$

Then there exists a sequence of operators $S_N : \mathcal{F}(M) \to \mathcal{F}(A_N)$, of norm less than 72, such that for every $\gamma \in \mathcal{F}(M)$ the sequence $(S_N(\gamma))_{N\in\mathbb{N}}$ converges to $\gamma$.

Now let $M$ be a countable and proper metric space. Since every closed ball is compact, the set $A_N$ is countable and compact, for every $N \in \mathbb{N}$. Thus Theorem 3.1 in [1] asserts that $\mathcal{F}(A_N)$ has the MAP and since for every $N \in \mathbb{N}$, $\mathcal{F}(A_N)$ is separable, there exists $R_p^N : \mathcal{F}(A_N) \to \mathcal{F}(A_N)$ a sequence of operators of finite-rank, so that for every $\gamma \in \mathcal{F}(A_N)$, $\lim\limits_{p\to+\infty} R_p^N \gamma = \gamma$ and $\|R_p^N\| \leq 1$ for every $p \in \mathbb{N}$ ([18], see also Theorem 1.e.13 in [15]).

Setting $Q_{N,p} = R_p^N \circ S_N$ we deduce that the range of $Q_{N,p}$ is finite dimensional, $\|Q_{N,p}\| \leq \|R_p^N\|\|S_N\| \leq 72$ and for every $\gamma \in \mathcal{F}(M)$,

$$\lim_{N\to+\infty} \lim_{p\to+\infty} R_p^N S_N \gamma = \lim_{N\to+\infty} S_N \gamma = \gamma.$$

Thus $\mathcal{F}(M)$ has the 72-BAP.

Finally, we can conclude that $\mathcal{F}(M)$ has the MAP. □

## 3. Ultrametric spaces

A metric space $(M, d)$ is said to be ultrametric if for every $x, y, z \in M$, we have $d(x, z) \leq \max\{d(x, y), d(y, z)\}$. One can easily prove the following useful properties:

*Property* 1. For $x, y \in M$ and $r, r' > 0$, if $B(x, r) \cap B(y, r') \neq \emptyset$ and $r \leq r'$ then $B(x, r) \subset B(y, r')$.

*Property* 2. For $x, y \in M$ and $r > 0$, if $y \in B(x, r)$, then $B(y, r) = B(x, r)$.

*Property* 3. For $x, y, z \in M$, if $d(x, y) \neq d(y, z)$ then

$$d(x, z) = \max\{d(x, y), d(y, z)\}.$$

*Property* 4. For every $r > 0$ there exists a partition of $M$ in closed balls of radius $r$.

Now let us prove the first result of this section:

*Theorem* 3.1. The Lipschitz-free space over a proper ultrametric space has the metric approximation property.

*Proof:* Let $M$ be a proper ultrametric space and $\tau_p$ the topology of pointwise convergence on $Lip_0(M)$. We will construct a sequence $(L_n)_{n\in\mathbb{N}}$ of operators on $Lip_0(M)$, of norm less than 1, such that for every $f \in Lip_0(M)$ the sequence $(L_n f)_{n\in\mathbb{N}}$ converges pointwise to $f$.

Let $n \in \mathbb{N}$. Because $M$ is ultrametric there exists a partition of $\overline{B}(0, n)$ into balls $\overline{B}(x, \frac{1}{n})$. Moreover, the closed ball $\overline{B}\,(0, n)$ is compact then it is possible to find

$x_1, \cdots, x_k \in M$ such that $\{\overline{B}\left(x_i, \frac{1}{n}\right)\}_{i=1}^k$ is a finite partition of $\overline{B}(0,n)$. Now define $L_n : Lip_0(M) \to Lip_0(M)$ as follows:

$$\forall f \in Lip_0(M),\ L_n(f) : M \to \mathbb{R}$$
$$x \mapsto \begin{cases} f(x_i) & , \text{ where } x \in \overline{B}\left(x_i, \frac{1}{n}\right), 1 \leq i \leq k \\ 0 & , x \notin \overline{B}\left(0, n\right) \end{cases}$$

We will first compute the norm of $L_n$. Let $f \in Lip_0(M)$ and $x, y \in M$.

- If there exists $i \in \{1, \cdots, k\}$ such that $x, y \in \overline{B}\left(x_i, \frac{1}{n}\right)$ then clearly:
$$|L_n(f)(x) - L_n(f)(y)| = 0 \leq \|f\|_L d(x,y).$$
- Now assume $x \in \overline{B}\left(x_i, \frac{1}{n}\right)$ and $y \in \overline{B}\left(x_j, \frac{1}{n}\right)$ with $i \neq j$. Remark that because $x \in \overline{B}\left(x_i, \frac{1}{n}\right)$, we have $\overline{B}\left(x_i, \frac{1}{n}\right) = \overline{B}\left(x, \frac{1}{n}\right)$. Furthermore $y \notin \overline{B}\left(x_i, \frac{1}{n}\right) = \overline{B}\left(x, \frac{1}{n}\right)$, so $d(x,y) > \frac{1}{n}$.
$$\begin{aligned} |L_n(f)(x) - L_n(f)(y)| &= |f(x_i) - f(x_j)| \leq \|f\|_L d(x_i, x_j) \\ &\leq \|f\|_L \max\{d(x_i,x), d(x_j,x)\} = \|f\|_L d(x_j, x) \\ &\leq \|f\|_L \max\{d(x_j,y), d(y,x)\} = \|f\|_L d(x,y) \end{aligned}$$
- Finally, for $x \in \overline{B}(0,n)$ and $y \notin \overline{B}(0,n)$, there exists $i \in \{1, \cdots, k\}$ such that $x \in \overline{B}\left(x_i, \frac{1}{n}\right)$. Because $x \in \overline{B}(0,n)$, we have $\overline{B}(0,n) = \overline{B}(x,n)$ and since $y \notin \overline{B}(0,n)$, we obtain $d(x,y) > n$. Hence
$$|L_n(f)(x) - L_n(f)(y)| = |f(x_i)| \leq \|f\|_L d(x_i, 0) \leq \|f\|_L \times n \leq \|f\|_L d(x,y).$$

Then $\|L_n(f)\| \leq \|f\|_L$ and $\|L_n\| \leq 1$.

One can easily prove that $L_n$ is $\tau_p - \tau_p$-continuous and that for $f \in Lip_0(M)$, the sequence $(L_n(f))_{n\in\mathbb{N}}$ pointwise converges to $f$. Then it is the adjoint of an operator $R_n : \mathcal{F}(M) \to \mathcal{F}(M)$ of norm less than 1 such that for every $\gamma \in \mathcal{F}(M)$, the sequence $(R_n(\gamma))_{n\in\mathbb{N}}$ weakly-converges to $\gamma$. Finally, for every $n \in \mathbb{N}$ we have $R_n\left(\mathcal{F}(M)\right) = \text{span}\{\delta_{x_i}, 1 \leq i \leq k\}$, so the operator $R_n$ is of finite rank. Then, because $\mathcal{F}(M)$ is separable, using convex combinations and a diagonal argument, we can conclude that $\mathcal{F}(M)$ has the MAP [2]. $\square$

It is also possible to prove that the Lipschitz-free space over a proper ultrametric space $M$ is a dual space. We will prove first that in the case of $K$ a compact ultrametric space, $\mathcal{F}(K)$ is isometrically isomorphic to $lip_0(K)^*$. We will again use the result of Petunīn and Plīčko. Note that Theorem 3.3.3 in [20] provides an alternative approach.

Before stating the result let us introduce the notion of $\mathbb{R}$-trees and some background about its link with ultrametric spaces:

**Definition 3.2.** A metric space $(T, d)$ is said to be an $\mathbb{R}$-tree when the two following conditions hold:

(1) for every $a, b$ in $T$, there exists a unique isometry $\phi : [0, d(a,b)] \to T$ such that $\phi(0) = a$ and $\phi(d(a,b)) = b$.
(2) any continuous and one-to-one mapping $\varphi : [0,1] \to T$ has same range as the isometry $\phi$ associated to the points $a = \varphi(0)$ and $b = \varphi(1)$.

*Background.* P. Buneman proved in [3] that the 4-points property is a characterization of subsets of $\mathbb{R}$-trees, where a metric space $(M, d)$ has the 4-points property if for every $x, y, z$ and $t$ in $M$ we have:

$$d(x, y) + d(z, t) \leq \max \{d(x, z) + d(y, t) \ , \ d(x, t) + d(y, z)\} .$$

In particular any ultrametric space $(M, d)$ has the 4-points property.

It is proved in [16] by Matoušek that, for a subspace $M$ of a tree $T$, it is possible to find a linear extension operator from $Lip_0(M)$ to $Lip_0(T)$ which is bounded. In particular $\mathcal{F}(M)$ is complemented in $\mathcal{F}(T)$.

Moreover, Godard proved in [4] that the Lipschitz-free space over an $\mathbb{R}$-tree is an $L_1$-space.

In conclusion if $M$ is a ultrametric space, its Lipschitz-free space is complemented into an $L_1$-space.

*Theorem* 3.3. If $(K, d)$ is a compact ultrametric space, then $\mathcal{F}(K)$ is isometrically isomorphic to $lip_0(K)^*$ and $lip_0(K)$ is isomorphic to $c_0(\mathbb{N})$.

*Proof:* It is proved in [1] that for a compact metric space $(K, d)$, the space $lip_0(K)$ is a subset of $NA\,(\mathcal{F}(K))$ and it is separating as soon as it separates points uniformly.

Let $(K, d)$ be a compact ultrametric space. To obtain the first part of the result it is enough to prove that $lip_0(K)$ separates points uniformly.
Let $x, y \in K$, set $a = d(x, y)$ and define $h : K \to \mathbb{R}$ as follows:

$$\forall z \in K, \ h(z) = d(x, y) \left(\mathbf{1}_{B(x,a/2)}(z) - \mathbf{1}_{B(x,a/2)}(0)\right)$$

where $\mathbf{1}_{B(x,a/2)}$ is the characteristic function of the open ball $B(x, a/2)$.
Then we have $h(0) = 0$ and $|h(x) - h(y)| = d(x, y)$. We will compute the Lipschitz-constant of $h$:
If $z, t$ are both in $B(x, a/2)$ or both outside $B(x, a/2)$, then

$$|h(z) - h(t)| = 0 \leq 2d(z, t).$$

Take $z \in B(x, a/2)$ and $t \notin B(x, a/2)$, then

$$d(z, t) = \max\{d(x, z), d(x, t)\} = d(x, t) \geq \frac{a}{2} = \frac{d(x, y)}{2} = \frac{|h(z) - h(t)|}{2}.$$

Hence the function $h$ is 2-Lipschitz.

To conclude we need to prove that $h \in lip_0(K)$. We will see that $\delta = \frac{a}{2}$ holds for every $\varepsilon$:
Let $z, t \in K$ such that $d(z, t) < \frac{a}{2}$.
First, if $z \in B(x, a/2)$ then $B(x, a/2) = B(z, a/2)$ and because $d(z, t) < \frac{a}{2}$ we have $t \in B(x, a/2)$ and $h(z) = h(t)$.
Secondly, if $z \notin B(x, a/2)$ then $t$ cannot be in $B(x, a/2)$ and $h(z) = h(t)$.

This proves that $h$ is in $lip_0(K)$ so this space separates points uniformly and therefore this concludes the proof of the fact that $\mathcal{F}(K)$ is the dual space of $lip_0(K)$.

A result due to D.R. Lewis and C. Stegall [14] asserts that if a separable dual space is complemented in $L_1$, then it is isomorphic to $\ell_1(\mathbb{N})$. So it follows from the background before the theorem that $\mathcal{F}(K)$ is isomorphic to $\ell_1(\mathbb{N})$.

Finally, Theorem 6.6 in [11] asserts that for a compact metric space $K$, the space $lip_0(K)$ is isomorphic to a subspace of $c_0(\mathbb{N})$. Moreover, its dual is isomorphic to $\ell_1(\mathbb{N})$, then Corollary 2 in [10] implies that $lip_0(K)$ is isomorphic to $c_0(\mathbb{N})$. $\square$

More generally for a proper ultrametric space we have the following:

*Theorem* 3.4. Let $M$ be a proper ultrametric space and

$$S = \left\{ f \in lip_0(M);\ \lim_{r\to+\infty} \sup_{\substack{x \text{ or } y \notin \overline{B}(0,r) \\ x\neq y}} \frac{f(x)-f(y)}{d(x,y)} = 0 \right\}.$$

Then $\mathcal{F}(M)$ is isometrically isomorphic to $S^*$ and $S$ is isomorphic to $c_0(\mathbb{N})$.

*Proof:* It is possible to adapt the proof of Theorem 6.6 in [11] to obtain that the space $S$ is isomorphic to a subspace of $c_0(\mathbb{N})$:

*Lemma* 3.5. Let $M$ be a proper metric space. Then for any $\varepsilon > 0$, the space $S$ is $(1+\varepsilon)$-isometric to a subspace of $c_0(\mathbb{N})$.

*Proof:* Assume $\varepsilon < 1$ and consider the space $M \times M$ with the metric :

$$d((x_1,x_2),(y_1,y_2)) = \max\left\{d(x_1,y_1), d(x_2,y_2)\right\}.$$

For every $j \in \mathbb{N}$ and $k \in \mathbb{Z}$ we consider the compact set

$$C_{j,k} = \left\{(x_1,x_2) \in M \times M\ ;\ d(0,x_1) \leq 2^j \text{ and } \ 2^k \leq d(x_1,x_2) \leq 2^{k+1}\right\}$$

and $F_{j,k}$ a finite $2^{k-3}\varepsilon$-net of $C_{j,k}$. Then $F := \bigcup_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}} F_{j,k}$ is countable.

We now define

$$\begin{array}{rcl} T : S & \to & c_0(F) \\ f & \mapsto & \left(\dfrac{f(x_1)-f(x_2)}{d(x_1,x_2)}\right)_{(x_1,x_2)\in F} \end{array}.$$

Justify first that $Tf \in c_0(F)$ for $f \in S$:

Let $\alpha > 0$.

Because $f \in S$, in particular $f \in lip_0(M)$ and there exists $K \in \mathbb{N}$ such that for every $k \leq -K$, if $d(x_1,x_2) \leq 2^{k+1}$ then $\dfrac{|f(x_1)-f(x_2)|}{d(x_1,x_2)} \leq \alpha$. Thus for every $j \in \mathbb{N}$, every $k \leq -K$ and $(x_1,x_2) \in C_{j,k}$, we have $\dfrac{|f(x_1)-f(x_2)|}{d(x_1,x_2)} \leq \alpha$.

Moreover, $\lim_{r\to+\infty} \sup_{\substack{x \text{ or } y \notin \overline{B}(0,r) \\ x\neq y}} \dfrac{f(x)-f(y)}{d(x,y)} = 0$, thus there exists $R > 0$ such that $\forall r \geq R$, $\forall x \notin \overline{B}(0,r)$, $y \in M$, we have $\dfrac{|f(x)-f(y)|}{d(x,y)} \leq \alpha$.

Let $N \in \mathbb{N}$ be such that $2^n \geq 2R$, $\forall n \geq N$.

If $(x_1,x_2) \in C_{j,k}$ with $j \geq N$ we clearly have $\dfrac{|f(x_1)-f(x_2)|}{d(x_1,x_2)} \leq \alpha$

Assume now $(x_1,x_2) \in C_{j,k}$ with $k > N$ and $j \leq N$, then

$$d(0,x_2) \geq d(x_1,x_2) - d(0,x_1) \geq 2^k - R > R$$

that is $x_2 \notin \overline{B}(0,R)$ and $\dfrac{|f(x_1)-f(x_2)|}{d(x_1,x_2)} \leq \alpha$.

Finally, we obtain that $Tf \in c_0(F)$, for every $f \in S$.

Clearly $\|T\| \leq 1$. We will now show that $\|f\|_L \leq (1+\varepsilon)\|Tf\|_\infty$:
Let $y_1 \neq y_2 \in M$. There exists $j \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that $(y_1, y_2) \in C_{j,k}$ and $(x_1, x_2) \in F_{j,k}$ such that $d((y_1, y_2), (x_1, x_2)) \leq 2^{k-3}\varepsilon$. Then

$$\begin{aligned} d(y_1, y_2) &\geq d(x_1, x_2) - d(x_1, y_1) - d(x_2, y_2) \geq d(x_1, x_2) - 2^{k-2}\varepsilon \\ &\geq d(x_1, x_2)\left(1 - \frac{\varepsilon}{4}\right). \end{aligned}$$

Let $f \in S$,

$$\begin{aligned} \frac{|f(y_1) - f(y_2)|}{d(y_1, y_2)} &\leq \frac{|f(x_1) - f(x_2)|}{d(y_1, y_2)} + \frac{d(x_1, y_1) + d(x_2, y_2)}{d(y_1, y_2)}\|f\|_L \\ &\leq \frac{|f(x_1) - f(x_2)|}{d(y_1, y_2)} + \frac{\varepsilon}{4}\|f\|_L \\ &\leq \left(1 - \frac{\varepsilon}{4}\right)^{-1} \frac{|f(x_1) - f(x_2)|}{d(x_1, x_2)} + \frac{\varepsilon}{4}\|f\|_L \\ &\leq \left(1 - \frac{\varepsilon}{4}\right)^{-1} \|Tf\|_\infty + \frac{\varepsilon}{4}\|f\|_L \end{aligned}$$

Finally, $\|Tf\|_\infty \leq \|f\|_L \leq (1+\varepsilon)\|Tf\|_\infty$ and one can conclude that $S$ is $(1+\varepsilon)$-isometric to a subspace of $c_0(\mathbb{N})$. □

We now conclude the proof of Theorem 3.4. We previously proved that in the case of a proper metric space, the space $S$ is a subspace of $NA(\mathcal{F}(M))$ and it is separating as soon as it separates points uniformly. Therefore in order to use Petunīn and Plīčko's result [19] (see also [5]) we only need to prove that, in the case of proper ultrametric space, the space $S$ separates points uniformly.

For given $x, y \in M$, the function $h$ defined as in proof of Theorem 3.3 satisfies $h \in lip_0(M)$, $|h(x) - h(y)| = d(x, y)$ and its Lipschitz constant does not depend on $x$ and $y$.

Let $r > 0$ be such that $B(x, a/2) \subset B(0, r)$, with $a = d(x, y)$. We may and do assume that $d(z, 0) > r$.
First if $t \in \overline{B}(x, \frac{a}{2})$, then

$$\frac{|h(z) - h(t)|}{d(z, t)} = \frac{d(x, y)}{d(z, t)} \underset{r \to +\infty}{\longrightarrow} 0.$$

Secondly if $t \notin \overline{B}(x, \frac{a}{2})$, then

$$\frac{|f(z) - f(t)|}{d(z, t)} = 0.$$

Finally, we have $h \in S$, then $S$ separates points uniformly. We can conclude that $S^*$ is isometrically isomorphic to $\mathcal{F}(M)$.

The second part of the proof follows the same line than the last part of the proof of Theorem 3.3. □

*Remark* 3.6. B.R. Kloeckner proved in [12] that the Wasserstein space of a compact ultrametric space is affinely isometric to a convex subset of $\ell_1(\mathbb{N})$.

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